Fibred and Indexed Categories for Abstract Model Theory

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Abstract

Indexed and fibred category theory have a long tradition in computer science as a language to formalize different presentations of the notion of a logic, as for instance, in the theory of institutions and general logics, and as unifying models of (categorical) logic and type theory as well. Here we introduce the notions of indexed and fibred frames and construct a rich mathematical workspace where many relevant and useful concepts of logics can be elegantly modelled. To demonstrate the applicability of these tools, essential ideas around the theory of institutions are recasted and described.

Keywords: indexed and fibred categories, Grothendieck constructions, logical systems, institutions.

1 Introduction

The issue of providing a general or abstract definition for a well-known concept, and all of its many instances, can be taken as twofold regarding its motivation. One can use the abstract definition in order to obtain an important characterization result, such as it is the case for the Lindstrom’s theorem for elementary logic [17], or even to provide an abstract setting to deal with the many extensions of first-order logic and proving abstract results about the relationships holding between certain properties of these logics, as did the work of Barwise [2] and his followers. On the other hand, such general settings can be used to the general application of a known and successful technique to a circumscribed class of formal objects (logics, models or their integration), as it is the case when one obtains some of the meta-theorems on General Logics [20], or even the naive application of completion techniques to build canonical models in an abstract way, as one can observe in any textbook on Modal Logic and its Applications ([13, 25]). The use of Category Theory as a working tool in Logic has its origin in the Curry-Howard isomorphism [10] and provided fruitful ways of building models for substructural logics. On the other hand, the work of Makkai and Reyes on first-order and coherence logics [18] and the school of Elementary Topoi and foundations [21] provided fine details on dealing with basic issues on modelling as substitution and environments. Type Theory also provided examples (polymorphism, higher-order
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typed systems, etc) and methods (Proofs-as-Arrows, Internal and External logics, etc) of how the object of study can improve the metalanguage theory. Indexed and Fibred Categories as well as two and higher-dimensional categories are good example of the last mentioned phenomena.

On the other hand, the integration of the categorical techniques and the abstract setting for dealing with logics and algebraic languages, as well as the need for a wider framework to cope with the use and reuse of distinct formalisms in the process of software development and validation, produced the concept of Institution [11, 6, 5]. In this way, the theory of Institutions is then an abstract setting for Logics and Specifications, built on top of Category Theory basic concepts and it figures out the main mechanisms for a smooth integration of different logics. Here smooth integration means “almost meta-properties preserving” integration. Institutions and General Logics have provided also important means to explain the mechanism of borrowing theorems from one logic to be reused by another [7] and of models for unification of theories by way of adjunctions [1].

From the strict categorical point of view, indexed and fibred categories [27, 15] provide us with two opposite, yet complementary views to understand and formalize the rich set of notions present in any logical system. The connection of indexed categories and logic can be seen as an attempt to provide structural foundations for computer science in several fields. This can be related with first uses of universal algebra (via heterogeneous data types) and algebraic many-sorted specifications as foundational concepts for software engineering. Later, generalizations of this structural point of view naturally led to the ideas of Institutions and Heterogeneous Logical Systems as we know it today [11, 23, 8]. Fibred category theory, however, have been mostly helpful in order to deal with (unifying) connections between LOGIC AND TYPE THEORY [15]. So, in a certain way, fibred categories arises as the categorical way to use mathematical logic by way of logical properties (theories) as a foundational tool in computer science. Indeed, this kind of duality is already present in the more traditional and well known notion of Galois Connection.

To illustrate such claims, we recall that Equational Logic is a well-known Institution and it has been quite useful to specify data-types. In the literature there is a relevant amount of examples of typical data-types specifications in equational logic. Consider a simple programming language PL consisting of assignments, sequencing of commands, selection (if) and a while command. Apart from introducing structured data-types, the language semantics is naturally (in the categorical sense) invariant under simple data-types indexing. Specifically, there is a PL(bool) language when considering the programming languages variables ranging over booleans, there is another PL(Int) (PL(Float)) language when these variables range over float numbers or integers and so on. Technically speaking, there is an algebra for expressions (right-hand side of an assignment) for each data-type chosen, thus seeing PL(Σ) as a parametric programming language. Taking into account the existing morphisms between the respective data-types Float and Int, for instance, there is a mapping from programs in PL(Float) into programs in PL(Int) that lifts to a natural transformation regarding their respective semantics. This is a mechanism quite well-known from (parametric) polymorphic types semantics. For type-theoreticians this is a typical example of the use of Indexed or Fibred Categories. So to say, the Syntax and the Semantics of programming language are indexed by categories of the (scalar) data-
types. This motivates the introduction of Indexed Frames allowing the definition of Indexed Institutional Frames, an indexed, complete categorical view of the concept of an Institution.

Consider again the programming language $P_L(-)$. First-order logic (FOL) is also known to be an Institution. Consider a Hoare Triples definition for the semantics of $P_L(-)$. Of course, the indexing provided to $P_L(-)$ should be also provided in a natural way to FOL, such that each data-type is mapped a corresponding first-order signature that indexes the first-order theory logically related to the data-type algebra. Following this example, the Hoare Calculus can be better viewed as an Indexed or Fibred construction built up on top of another Indexed construction. It is known that under certain assumptions, Indexed and Fibred constructions are equivalent, but as already pointed out by [15] and [26] the Fibred approach is better suited to deal with modular constructions, as above. This kind of application motivates the Fibred Frame notion that is introduced in this article. It is worthwhile to mention that there are known examples pointing out that the introduction of structured data-types, namely, arrays, records, graphs and so on, would modify the structure of the semantics assignment to a programming language. So to say, $P_L$ extended with these structured data-types would not have their semantics provided by Hoare calculus instances that are parametric. One has to hard wire for each structured data-type a set of essentially distinct Hoare rules. However, in any case there should be a natural relationship between syntax and semantics regarding each Fiber induced by each $P_L$ extension.

One of our overall research interests is to explore and to investigate the potential room of possible concepts for presenting and investigating logics and their relationships. This paper is devoted to elucidate the equivalence of indexed and fibred categories in view of logics and specification formalisms. We propose two new clean abstract notions of “logical systems” – Indexed Frames and Split Fibred Frames. We show the equivalence of both concepts and we give some evidence that these concepts are reasonable and appropriate for studying logics and their relationships. A thorough analysis of the concept of Institution shows that the satisfaction condition can be equivalently expressed by a naturality requirement if we abstract from single sentences and single models to specifications and model classes, respectively. This observation extends to a construction that lifts any institution to a corresponding Indexed Institutional Frame (and thus also to a corresponding Fibred Institutional Frame).

### 1.1 Notations and Basics

With respect to notation, the collection of objects of a category $C$ will be denoted $|C|$. Given objects $a, b \in |C|$, the collection (usually a set) of arrows from $a$ to $b$ is denote $C(a, b)$. The category $\text{Set}$ is the category of sets and total functions, while $\text{Cat}$ is the category of all categories and functors.

We represent composition of maps (functors) in diagrammatic order. For instance if $F : A \to B$ and $G : B \to C$ are functors and $a \in |A|$, then $(F; G)(a) \overset{\text{def}}{=} G(F(a))$ is an object of $C$, i.e., $G(F(a)) \in |C|$.

Also, different institutions, frames, logics, are identified with primed superscripts (e.g., $I, I', I'', \text{etc.}$), while different objects within an institution (frame, logic), as sig-
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natures, models, etc., are denoted with numbered subscripts (e.g., $\Sigma_1, \Sigma_2, M_1, M_2$, etc.).

Moreover, if $\alpha : F \Rightarrow G : A \rightarrow B$ and $\beta : G \Rightarrow H : A \rightarrow B$ are natural transformations, then the vertical composition of $\alpha$ and $\beta$ is denoted $\alpha;\beta : F \Rightarrow H : A \rightarrow B$ such that for each $a \in |A|$, $(\alpha;\beta)_a \overset{def}{=} \alpha_a;\beta_a$. Also, if $F : A \rightarrow B$, $G, G' : B \rightarrow C$, $H : C \rightarrow D$ are functors and $\alpha : G \Rightarrow G' : B \rightarrow C$ is a natural transformation, then the horizontal compositions of $F$ with $\alpha$, and $\alpha$ with $H$ are represented as $(F \cdot \alpha) : F; G \Rightarrow F; G' : A \rightarrow C$ and $(\alpha \cdot H) : G; H \Rightarrow G'; H : B \rightarrow D$ such that for each $c \in |C|, a \in |A|, (F \cdot \alpha)_c \overset{def}{=} \alpha_{F(c)}$ whereas $(\alpha \cdot H)_a \overset{def}{=} H(\alpha_a)$.

\[
\begin{array}{c}
A \xrightarrow{F} B \xrightarrow{G} C \xrightarrow{H} D
\end{array}
\]

We have immediately according the definition of composition

\[id_B \cdot \alpha = \alpha \quad \text{and} \quad \alpha \cdot id_C = \alpha \quad (1.1)\]

and since functors map identities to identities we have also

\[F \cdot id_G = id_{F,G} \quad \text{and} \quad id_G \cdot H = id_{G,H} \quad (1.2)\]

Moreover we have for functors $F : A \rightarrow B$, $G : B \rightarrow C$, $H, H', H'' : C \rightarrow D$, $I : D \rightarrow E$ and natural transformations $\gamma : H \Rightarrow H'$, $\delta : H' \Rightarrow H''$ the following laws

\[
\begin{array}{c}
A \xrightarrow{F} B \xrightarrow{G} C \xleftarrow{\gamma} D \xrightarrow{\delta} E
\end{array}
\]

\[\begin{align*}
(F; G) \cdot \gamma &= F \cdot (G \cdot \gamma) & \text{and} & & G \cdot (\gamma; \delta) &= (G \cdot \gamma); (G \cdot \delta) \\
(G \cdot \gamma) \cdot I &= G \cdot (\gamma \cdot I) & \text{and} & & (\gamma; \delta) \cdot I &= (\gamma \cdot I); (\delta \cdot I)
\end{align*}\]

(1.3) (1.4)

2 Indexed Logical Systems

2.1 Indexed Frames

**Definition 2.1 (Indexed Functor)**

Let $F : \text{Ind} \rightarrow \text{Cat}$ and $G : \text{Ind} \rightarrow \text{Cat}$ be Ind-indexed categories. Then an Ind-indexed functor is a natural transformation $\alpha : F \Rightarrow G : \text{Ind} \rightarrow \text{Cat}$, such that for each $\sigma : i \rightarrow j$ in Ind the following diagram commutes:

\[
\begin{array}{ccc}
i & \xrightarrow{F(i)} & G(i) \\
\sigma & \downarrow F(\sigma) & \downarrow G(\sigma) \\
j & \xrightarrow{F(j)} & G(j)
\end{array}
\]

The category of Ind-indexed categories and Ind-indexed functors will be denoted $\text{IndFun}(\text{Ind}, \text{Cat})$.  

\[\blacksquare\]
DEFINITION 2.2 (Indexed Frame)
An Indexed Frame

\[ \mathcal{IF} = (\text{Ind}, \text{Syn}, \text{Den}, \text{sem}, \text{fml}) \]

is given by the following data and operations:

\[ \text{A category } \text{Ind}; \]
\[ \text{An indexed category } \text{Syn} : \text{Ind} \rightarrow \text{Cat}. \]
\[ \text{An indexed category } \text{Den} : \text{Ind} \rightarrow \text{Cat}. \]
\[ \text{An indexed functor } \text{sem} : \text{Syn} \Rightarrow \text{Den} : \text{Ind} \rightarrow \text{Cat}, \text{ that is to say, for each } i \in |\text{Ind}|, \text{ we have a functor } \text{sem}(i) : \text{Syn}(i) \rightarrow \text{Den}(i) \text{ such that for each } \sigma : i \rightarrow j \text{ in } \text{Ind}, \text{ } \text{sem}(i) ; \text{Den}(\sigma) = \text{Syn}(\sigma) ; \text{sem}(j); \]
\[ \text{for each } i \in |\text{Ind}| \text{ a functor } \text{fml}(i) \text{ which is locally left-adjoint to } \text{sem}(i), \text{ i.e., } \text{fml}(i) \dashv \text{sem}(i). \]

\[ \begin{array}{ccc}
\text{Ind} & \xrightarrow{\text{Syn}} & \text{Cat} \\
\text{Den} & \xrightarrow{\text{sem}} & \\
\end{array} \]

\[ \begin{array}{ccc}
\text{Syn}(i) & \xleftarrow{\text{fml}(i)} & \text{Den}(i) \\
\downarrow{\text{sem}(i)} & & \downarrow{\text{Den}(\sigma)} \\
\text{Syn}(\sigma) & \xleftarrow{\text{sem}(\sigma)} & \text{Den}(\sigma) \\
\end{array} \]

Remark 2.3 (Interpretation)
In the context of this work, \( \text{Ind} \) can be understood as representing the syntactical entities (presentation) for the generation of the formulae of a given logical system. \( \mathcal{L}, \text{ Syn and Den} \) denote, respectively, the specifications and semantics of \( \mathcal{L} \), while \( \text{fml} \) (syntax) and \( \text{sem} \) (semantic models) capture the fact that syntax and semantics are always adjoint to each other (Galois Connection) via a concept of satisfaction or validity.

It is quite natural to require that a map between indexed frames should be a compatible parallel translation of syntax and semantics, respectively:

DEFINITION 2.4 (Maps of Indexed Frames)
Let \( \mathcal{IF} = (\text{Ind}, \text{Syn}, \text{Den}, \text{sem}, \text{fml}) \) and \( \mathcal{IF}' = (\text{Ind}', \text{Syn}', \text{Den}', \text{sem}', \text{fml}') \) be indexed frames. A map \( \xi = (\Upsilon, \gamma, \delta) : \mathcal{IF} \rightarrow \mathcal{IF}' \) between indexed frames is comprised of the following data and operations:

\[ \text{A functor } \Upsilon : \text{Ind} \rightarrow \text{Ind'} \]
\[ \text{A natural transformation } \gamma : \text{Den} \Rightarrow \Upsilon ; \text{Den}' : \text{Ind} \rightarrow \text{Cat}, \text{ and} \]
\[ \text{A natural transformation } \delta : \text{Syn} \Rightarrow \Upsilon ; \text{Syn}' : \text{Ind} \rightarrow \text{Cat}, \]
such that \( \text{sem}; \gamma = \delta; \text{sem}' \).

Due to the clean categorical simplicity of our definitions we obtain a

**Definition 2.5 (Category of Indexed Frames)**

Indexed frames and Maps of indexed frames define a category \( \text{IndFra} \). The identities \( \text{id}_{\mathcal{IF}} : \mathcal{IF} \rightarrow \mathcal{IF} \) are given by \( \text{id}_{\mathcal{IF}} \equiv (\text{id}_{\text{Int}}, \text{id}_{\text{Den}}, \text{id}_{\text{Syn}}) \). And, given maps \( \xi = (\Upsilon, \gamma, \delta) : \mathcal{IF} \rightarrow \mathcal{IF}' \) and \( \xi' = (\Upsilon', \gamma', \delta') : \mathcal{IF}' \rightarrow \mathcal{IF}'' \) the composition \( \xi; \xi' : \mathcal{IF} \rightarrow \mathcal{IF}'' \) is defined as follows:

\[
\xi; \xi' \equiv (\Upsilon; \Upsilon', \gamma; (\Upsilon \cdot \gamma'), \delta; (\Upsilon \cdot \delta')) : \mathcal{IF} \rightarrow \mathcal{IF}''.
\]

**Proof.** Neutrality: We obtain according to the definition of identities and composition, and due to 1.1

\[
\text{id}_{\mathcal{IF}}; \xi = (\text{id}_{\text{Int}}; \Upsilon, \text{id}_{\text{Den}}; \gamma, \text{id}_{\text{Syn}}; \delta) = (\Upsilon, \gamma, \delta) = \xi.
\]

And taking into account 1.1 we obtain

\[
\xi; \text{id}_{\mathcal{IF}} = (\Upsilon; \text{id}_{\text{Int}}, \gamma; (\Upsilon \cdot \text{id}_{\text{Den}}), \text{id}_{\text{Syn}}; \delta(\Upsilon \cdot \text{id}_{\text{Syn}})) = (\Upsilon, \gamma, \delta)
\]

Associativity: Let be given maps \( \xi = (\Upsilon, \gamma, \delta) : \mathcal{IF} \rightarrow \mathcal{IF}' \), \( \xi' = (\Upsilon', \gamma', \delta') : \mathcal{IF}' \rightarrow \mathcal{IF}'' \), and \( \xi'' = (\Upsilon'', \gamma'', \delta'') : \mathcal{IF}'' \rightarrow \mathcal{IF}''' \). We have to show \( (\xi; \xi') \circ \xi'' = \xi \circ (\xi'; \xi'') \).

For the first component we have \( (\Upsilon; \Upsilon'); \Upsilon'' = \Upsilon; (\Upsilon'; \Upsilon'') \) since the composition of functors is associative.
For the second composition we obtain the required equation due to the associativity
of the composition of functors and of the vertical composition of natural transforma-
tions, and due to 1.3

\[(\gamma; (\Upsilon . \gamma')); ((\Upsilon; \Upsilon') . \gamma'')\]
\[= \gamma; ((\Upsilon; \gamma'); ((\Upsilon; \Upsilon') . \gamma''))\]
\[= \gamma; ((\Upsilon; \gamma'); (\Upsilon; \gamma''))\]

The required equation for the third component is obtained in the same way.

2.2 Institutions

The concept of an institution introduced by Goguen and Burstall [11] formally
captures the notion of logical systems and allowed them to reformulate and to gen-
eralize the work they had done in the 70’s on structuring (equational) specification
independently of the underlying logic. A similar proposal of an abstract concept of
a logic had been given already by Barwise [2].

Definition 2.6 (Institution)

An Institution \( I = (\text{Sign}, \text{Sen}, \text{Mod}, \models) \) consists of the following data and operations:

- A category of abstract signatures \( \text{Sign} \);
- A syntax functor \( \text{Sen} : \text{Sign} \rightarrow \text{Set} \), defining for each signature its set of
  sentences;
- A model functor \( \text{Mod} : \text{Sign}^{op} \rightarrow \text{Cat} \), defining for each signature, its corre-
sponding category of models;
- An indexed family of satisfaction relations \( \models \subseteq |\text{Mod}(\Sigma)| \times \text{Sen}(\Sigma), \Sigma \in |
\text{Sign}| \), such that the following institution condition

\[ \text{Mod}(\phi)(M_2) \models_\Sigma \varphi_1 \iff M_2 \models_\Sigma_2 \text{Sen}(\phi)(\varphi_1) \]

holds for each \( \phi : \Sigma_1 \rightarrow \Sigma_1 \) in \( \text{Sign} \), \( M_2 \in |\text{Mod}(\Sigma_2)| \), and \( \varphi_1 \in \text{Sen}(\Sigma_1) \).

\[
\begin{array}{ccc}
\Sigma_1 & \xrightarrow{\text{Mod}(\Sigma_1)} & \text{Sen}(\Sigma_1) \\
\phi \downarrow & \downarrow \text{Sen}(\phi) & \\
\Sigma_2 & \xleftarrow{\text{Mod}(\Sigma_2)} & \text{Sen}(\Sigma_2)
\end{array}
\]

Remark 2.7 (Component Implications)

It will be essential for us to have a deeper understanding of the above institution
condition when we see it from the perspective of the two component implications.
Therefore, we label them for future reference.

\[
\begin{align*}
\text{IC}_{\text{Sen} \Rightarrow \text{Mod}} & \quad \text{Mod}(\phi)(M_2) \models_{\Sigma_1} \varphi_1 \iff M_2 \models_{\Sigma_2} \text{Sen}(\phi)(\varphi_1) \\
\text{IC}_{\text{Mod} \Rightarrow \text{Sen}} & \quad \text{Mod}(\phi)(M_2) \models_{\Sigma_1} \varphi_1 \Rightarrow M_2 \models_{\Sigma_2} \text{Sen}(\phi)(\varphi_1)
\end{align*}
\]
To keep the exposition short and accessible for a broader audience we concentrate on two well-known examples of institutions – equational logic and many-sorted equational logic. For more elaborated examples we refer to the literature [11, 19, 20, 24, 28, 29].

**Example 2.8 (Institution $\mathcal{EQ}$ of Equational Logic)**

This is the classical unsorted universal algebra and goes back to Birkhoff. Signatures $\Sigma = (OP, ar)$ are given by a set $OP$ of operation symbols and an arity function $ar : OP \to \mathbb{N}^*$. Signature morphisms $\phi : \Sigma_1 \to \Sigma_2$ are given by a map $\phi : OP_1 \to OP_2$ such that $ar_2(\phi(\alpha)) = ar_1(\alpha)$ for every $\alpha \in OP_1$. Signatures and signature morphisms form the category of signatures $\text{Sign}_\mathcal{EQ}$.

For any signature $\Sigma$ and any set $X$ of variables we define inductively the set $T(\Sigma, X)$ of $\Sigma$-terms over $X$. This we can assign to any signature $\Sigma$ the set $\text{Sen}_\mathcal{EQ}(\Sigma)$ of $\Sigma$-equations $(l = r)$ where $l, r \in T(\Sigma, X)$. Every signature morphism $\phi : \Sigma_1 \to \Sigma_2$ induces inductively a translation $\phi : T(\Sigma_1, X) \to T(\Sigma_2, X)$ of $\Sigma_1$-terms over $X$ into $\Sigma_2$-terms over $X$ and provides a functor $\text{Sen}_\mathcal{EQ}(\phi) : \text{Sen}_\mathcal{EQ}(\Sigma_1) \to \text{Sen}_\mathcal{EQ}(\Sigma_2)$. The sets $\text{Sen}_\mathcal{EQ}(\Sigma)$ and the translation functors $\text{Sen}_\mathcal{EQ}(\phi)$ constitute the syntax functor $\text{Sen}_\mathcal{EQ} : \text{Sign}_\mathcal{EQ} \to \text{Set}$.

For any signature $\Sigma$ we define the category $\text{Mod}_\mathcal{EQ}(\Sigma)$ of $\Sigma$-algebras as follows. The objects are $\Sigma$-algebras $A$, i.e. there is a non-empty carrier set $A$ and for every $\alpha \in OP$, $ar(\alpha) = n$ there is an operation $A(\alpha) : A^n \to A$. The morphisms are $\Sigma$-homomorphisms $h : A \to B$ translating the carriers compatible with the operations, i.e. $b(A(\alpha)(a_1, \ldots, a_n)) = B(\alpha)(h(a_1), \ldots, h(a_n))$, for every $\alpha \in OP$ and $a_i \in A$.

Given a signature morphism $\phi : \Sigma_1 \to \Sigma_2$ any $\Sigma_2$-algebra $A_2$ defines a $\Sigma_1$-algebra $\text{Mod}_\mathcal{EQ}(A_2)$ with the same carrier $A_2$ and with $\text{Mod}_\mathcal{EQ}(A_2)(\alpha p_1) = A_2(\phi(\alpha)p_1))$ for every $\alpha p_1 \in OP_1$. This construction also applies to $\Sigma_2$-homomorphisms thus we obtain a forgetful functor $\text{Mod}_\mathcal{EQ}(\phi) : \text{Mod}_\mathcal{EQ}(\Sigma_2) \to \text{Mod}_\mathcal{EQ}(\Sigma_1)$. The categories $\text{Mod}_\mathcal{EQ}(\Sigma)$ and the forgetful functors $\text{Mod}_\mathcal{EQ}(\phi)$ constitute the model functor $\text{Mod}_\mathcal{EQ} : \text{Sign}_\mathcal{EQ}^{op} \to \text{Cat}$.

Given a signature $\Sigma$, a $\Sigma$-algebra $A$, and a set $X$ of variables any variable assignment $\alpha : X \to A$ can be extended inductively to a term evaluation $\overline{\alpha} : T(\Sigma, X) \to A$. A $\Sigma$-equation $(l = r)$ is satisfied in $A$, $A \models_\Sigma (l = r)$ in symbols, iff $\overline{\alpha}(l) = \overline{\alpha}(r)$ for all assignments $\alpha$.

Let be given a signature morphism $\phi : \Sigma_1 \to \Sigma_2$, a $\Sigma_1$-equation $(l = r)$, and a $\Sigma_2$-algebra $A_2$. The crucial technical result for proving the institution condition

$$\text{Mod}_\mathcal{EQ}(\phi)(A_2) \models_{\Sigma_1} (l = r) \iff A_2 \models_{\Sigma_2} (\phi(l) = \phi(r))$$

is that the assignments of $X$ into $\text{Mod}_\mathcal{EQ}(\phi)(A_2)$ coincide with the assignments of $X$ into $A_2$ and that we have $\overline{\alpha}(t) = \overline{\phi}(\overline{\alpha}(t))$ for any assignment $\alpha : X \to A_2$ and for any $t \in T(\Sigma_1, X)$ (compare [29]).

**Example 2.9 (Institution $\mathcal{MEQ}$ of Many-Sorted Equational Logic)**

Signatures are of the form $\Sigma = (S, OP, \text{dom}, \text{cod})$ with $S$ a set of sort symbols, $OP$ a set of operation symbols, the domain function $\text{dom} : OP \to S^*$, and the codomain function $\text{cod} : OP \to S$. Signature morphisms $\phi : \Sigma_1 \to \Sigma_2$ translate sort and operation symbols compatible with the domain and codomain functions, i.e. $\text{dom}_2(\phi(\alpha)) = \phi'(\text{dom}_1(\alpha))$ and $\text{cod}_2(\phi(\alpha)) = \phi(\text{cod}_1(\alpha))$ for every $\alpha \in OP_1$. Signatures and signature morphisms form the category of many-sorted signatures $\text{Sign}_\mathcal{MEQ}$. 
For any signature $\Sigma$ and any $S$-set $X = (X_s \mid s \in S)$ of variables we define inductively the $S$-set $T(\Sigma, X)$ of $\Sigma$-terms over $X$. Thus we can assign to any signature $\Sigma$ the set $\text{Sen}_{\text{MEQ}}(\Sigma)$ of $\Sigma$-equations $(X : l = r)$ where $l, r \in T(\Sigma, X)(s), s \in S$. Every signature morphism $\phi : \Sigma_1 \to \Sigma_2$ induces a translation of $S_1$-sets $X = (X(s_1) \mid s_1 \in S_1)$ of variables into $S_2$-sets $\phi(X)$ of variables where $\phi(X)(s_2) = \{\{X(s_1) \mid \phi(s_1) = s_2\}$ for any $s_2 \in S_2$. This translation extends inductively to a family $\phi_{S_1} : T(\Sigma_1, X)(s_1) \to T(\Sigma_2, \phi(X))(\phi(s_1)), s_1 \in S_1$ of translations of $\Sigma_1$-terms over $X$ into $\Sigma_2$-terms over $\phi(X)$ and provides, finally, a functor $\text{Sen}_{\text{MEQ}}(\phi) : \text{Sen}_{\text{MEQ}}(\Sigma_1) \to \text{Sen}_{\text{MEQ}}(\Sigma_2)$ where $\text{Sen}_{\text{MEQ}}(\phi)(X : l = r) = (\phi(X) : \phi(s_1)(l) = \phi(s_1)(r))$. The sets $\text{Sen}_{\text{MEQ}}(\Sigma)$ and the translation functors $\text{Sen}_{\text{MEQ}}(\phi)$ constitute the syntax functor $\text{Sen}_{\text{MEQ}} : \text{Sign}_{\text{MEQ}} \to \text{Set}$.

For any signature $\Sigma$ we define the category $\text{Mod}_{\text{MEQ}}(\Sigma)$ of $\Sigma$-algebras as follows. The objects are $\Sigma$-algebras $A$, i.e. for every $s \in S$ there is a (possibly empty) carrier set $A(s)$ and for every $op : s_1, \ldots, s_n \to s$ in $\text{OP}$ there is an operation $A(op) : A(s_1) \times \ldots \times A(s_n) \to A(s)$. The morphisms are $\Sigma$-homomorphisms $h = (h(s) : A(s) \to B(s) \mid s \in S)$ translating the carriers compatible with the operations, i.e. $h(s)(A(op)(a_1, \ldots, a_n)) = B(op)(h(s_1)(a_1), \ldots, h(s_n)(a_n))$, for every $op : s_1, \ldots, s_n \to s$ in $\text{OP}$ and $a_i \in A(s_i), i = 1, \ldots, n$.

A signature morphism $\phi : \Sigma_1 \to \Sigma_2$ defines an interpretation of the components of $\Sigma_1$ by suitable components of $\Sigma_2$. Analogously a $\Sigma_2$-algebra $A_2$ is given by an interpretation of the components of $\Sigma_2$ by suitable components of $\text{Set}$. Composing these two interpretations we obtain an interpretation of $\Sigma_1$ in $\text{Set}$, i.e. a $\Sigma_1$-algebra $\text{Mod}_{\text{MEQ}}(A_2)$ with $\text{Mod}_{\text{MEQ}}(A_2)(s_1) = A_2(\phi(s_1))$ for every $s_1 \in S_1$ and with $\text{Mod}_{\text{MEQ}}(A_2)(op_1) = A_2(\phi(op_1))$ for every $op_1 \in \text{OP}_1$. This construction also applies to $\Sigma_2$-homomorphisms thus we obtain a forgetful functor $\text{Mod}_{\text{MEQ}}(\phi) : \text{Mod}_{\text{MEQ}}(\Sigma_2) \to \text{Mod}_{\text{MEQ}}(\Sigma_1)$. The categories $\text{Mod}_{\text{MEQ}}(\Sigma)$ and the forgetful functors $\text{Mod}_{\text{MEQ}}(\phi)$ constitute the model functor $\text{Mod}_{\text{MEQ}} : \text{Sign}_{\text{MEQ}} \to \text{Cat}$.

Given a signature $\Sigma$, a $\Sigma$-algebra $A$, and an $S$-set $X = (X_s \mid s \in S)$ of variables any variable $S$-assignment $\alpha = (\alpha(s) : X(s) \to A(s) \mid s \in S)$ can be extended inductively to a term evaluation $\overline{\alpha} = (\overline{\alpha}(s) : T(\Sigma, X)(s) \to A(s) \mid s \in S)$. A $\Sigma$-equation $(X : l = r)$ is satisfied in $A$, $A \models_\Sigma (X : l = r)$ in symbols, iff $\overline{\alpha}(s)(l) = \overline{\alpha}(s)(r)$ for all assignments $\alpha$. Note, that $(X : l = r)$ is vacuously satisfied if $X(s) \neq \emptyset$ and $A(s) = \emptyset$ for one $s \in S$.

Let be given a signature morphism $\phi : \Sigma_1 \to \Sigma_2$, a $\Sigma_1$-equation $(X : l = r)$, and a $\Sigma_2$-algebra $A_2$. The crucial technical result for proving the institution condition

$$\text{Mod}_{\text{MEQ}}(\phi)(A_2) \models_{\Sigma_1} (X : l = r) \iff A_2 \models_{\Sigma_2} (\phi(X) : \phi(s_1)(l) = \phi(s_1)(r))$$

is that there is a one-to-one correspondence between the assignments of $X$ into $\text{Mod}_{\text{MEQ}}(\phi)(A_2)$ and the assignments of $\phi(X)$ into $A_2$, respectively, and that there is a coincidence between the corresponding term evaluations. That is, for any $S_1$-assignment $\alpha : X \to \text{Mod}_{\text{MEQ}}(\phi)(A_2)$ there is a $S_2$-assignment $\beta : \phi(X) \to A_2$ defined by $\beta(\phi(s_1))(x) = \alpha(s_1)(x)$ for all $s_1 \in S_1$ and $x \in X(s_1)$ such that $\overline{\alpha}(s_1)(t) = \overline{\beta}(\phi(s_1))(\phi(s_1)(t))$ for all $t \in T(\Sigma_1, X)(s_1)$ [29].

There are three essential traditional features of logics reflected by the concept of institution: First, also in accordance with indexed frames, logics are described in a modular, stepwise manner: We fix a signature, i.e., the syntactical building blocks,
and then we construct the actual syntax for this fixed signature. And, these con-
structions are canonical in a way that they are compatible w.r.t. translations between
signatures. Moreover, we define what the models for a fixed signature are assumed to
be. Second, in contrast to indexed frames, syntax and semantics are connected on the
level of single sentences and single models by a corresponding concept of satisfaction.
Third, syntax and semantics behave contravariant w.r.t. translations. That is, any
signature morphism induces a translation of syntax in one direction and a respective
translation of semantics in the opposite direction.

Since institutions build on single sentences and models it is natural to define tran-
sitions between institutions also on this level. Moreover, the contravariance between
syntax and semantics will also appear in those transitions. The following concept is
equivalent to the original concept of “plain map of institution” in [20] (compare [19]),
but we adopt the up-to-date terminology from [12]. It formalizes how a target logic
I′ can code a source logic I. It requires that the syntax of
I′ is rich enough to define
the subclasses of models of
I′ which can be understood as models of I.

**Definition 2.10 (Institution Comorphisms)**

An **Institution Comorphism** \( \mu = (\Phi, \alpha, \beta) : I \to I′ \) consists of:

\[ \begin{aligned}
&\text{A functor } \Phi : \text{Sign} \to \text{Sign}′; \\
&\text{A natural transformation } \alpha : \text{Sen} \Rightarrow \Phi; \text{Sen}′ : \text{Sign} \to \text{Set}; \\
&\text{A natural transformation } \beta : \Phi^\text{op}; \text{Mod} \Rightarrow \text{Mod} : \text{Sign}^\text{op} \to \text{Cat}, \text{ such that } \\
&\text{the following comorphism condition:}
\end{aligned} \]

\[ \beta(\Sigma)(M′) \models \Sigma \varphi \iff M′ \models_{\Phi(\Sigma)} \alpha(\Sigma)(\varphi) \]

holds for each \( \Sigma \in | \text{Sign} |, \varphi \in \text{Sen}(\Sigma) \) and \( M′ \in | \text{Mod}′(\Phi(\Sigma)) | \).

**Example 2.11 (Institution Comorphism from \( MEQ \) into \( EQ \))**

The process of “omitting sorts” provides a comorphism from \( MEQ \) into \( EQ \). We map a many-sorted signature \( \Sigma = (S, OP, \text{dom, cod}) \) to the signature \((OP, ar)\) such that
\( ar(op) = n \) iff \( \text{dom}(op) \) has length \( n \). This defines a functor \( \Phi : \text{Sign}_{MEQ} \to \text{Sign}_{EQ} \).

For any many-sorted signature \( \Sigma \) and any \( S \)-set \( X = \{X_s \mid s \in S\} \) of variables we obtain the “unsorted” set \( \{X_s \mid s \in S\} = \{\langle s, x \rangle \mid s \in S, x \in X(s)\} \) of variables and for every \( s \in S \) we have an inclusion \( T(\Sigma, X)(s) \subseteq T(\Phi(\Sigma), \{X\}) \).

This means that we have a natural transformation \( \alpha : \text{Sen}_{MEQ} \Rightarrow \Phi; \text{Sen}_{EQ} \) with
\( \alpha(\Sigma)(X : l = r) = (l = r) \). Note, that in most cases \( \{\langle T(\Sigma, X)(s) \mid s \in S\} \) is a proper subset of \( T(\Phi(\Sigma), \{X\}) \) because many-sortedness means essentially, to put, in
addition to the arity constraints, further constraints on the construction of terms.

Moreover, we can associate to any \( \Phi(\Sigma) \)-algebra \( A \) a \( \Sigma \)-algebra \( \beta(\Sigma)(A) \) with car-
riers \( \beta(\Sigma)(A)(s) = A \) for all \( s \in S \) and operations \( \beta(\Sigma)(A)(op) = A(op) \) for all
This gives a functor $\beta(\Sigma) : \text{Mod}_{\mathcal{EQ}}(\Phi(\Sigma)) \to \text{Mod}_{\mathcal{MEQ}}(\Sigma)$, and globally defines a natural transformation $\beta : \Phi^\text{op} ; \text{Mod}_{\mathcal{EQ}} \Rightarrow \text{Mod}_{\mathcal{MEQ}}$.

In accordance with the institution conditions the comorphism condition

$$\beta(\Sigma)(A) \models_{\Sigma} (X : l = r) \iff A \models_{\Phi(\Sigma)} (l = r)$$

is due to a one-to-one correspondence between the assignments of $X$ into $\beta(\Sigma)(A)$ and the assignments of $\bigcup X$ into $A$, respectively, and due to the coincidence between the corresponding term evaluations. That is, for any $S$-assignment $\alpha : X \to \beta(\Sigma)(A)$ there is an assignment $\beta : \bigcup X \to A$ defined by $\beta(s,x) = \alpha(s)(x)$ for all $s \in S$, $x \in X(s)$ such that $\pi(s)(t) = \beta(t)$ for all $t \in T(\Sigma,X)(s)$.

**Definition 2.12 (Category of Institutions)**

Institutions and institution comorphisms define a category $\text{InstCom}$. The identities $\text{id}_I : I \to I$ are given by $\text{id}_I \overset{\text{def}}{=} (\text{id}_{\text{Sign}}, \text{id}_{\text{Sen}}, \text{id}_{\text{Mod}})$. And, given comorphisms $\mu = (\Phi, \alpha, \beta) : I \to I'$ and $\mu' = (\Phi', \alpha', \beta') : I' \to I''$, the composition $\mu ; \mu' : I \to I''$ is defined as follows:

$$\mu ; \mu' \overset{\text{def}}{=} (\Phi ; \Phi', \alpha ; (\Phi \cdot \alpha'), (\Phi^\text{op} ; \beta') ; \beta) : I \to I''.$$

**Proof.** Analogously to the proof of Definition 2.5.

### 2.3 Indexed Institutional Frames

Institutions are based on a pointwise assignment of signatures, sentences, and models. In design (programming), however, the relevant objects are not sentences (program lines), but, specifications (programs). Now we systematically reveal the categorical structures that are intrinsic to logical systems on this relevant level of specifications (and subcategories of models). Firstly, these investigations provide new insights into the conceptual nature of logical systems so that we can, for instance, give a very simple, although enlightening categorical description of the institution condition. Secondly, the derived structures establish special instances of indexed frames.

Assuming an institution $\mathcal{I}$, and a set of $\Sigma$-sentences $\Gamma \subseteq \text{Sen}(\Sigma)$, we define the category $\text{mod}(\Sigma)(\Gamma)$ as the full subcategory induced by those models in $|\text{Mod}(\Sigma)|$ that satisfy $\Gamma$, i.e.,

$$|\text{mod}(\Sigma)(\Gamma)| \overset{\text{def}}{=} \{ M \in |\text{Mod}(\Sigma)| \mid \forall \varphi \in \Gamma : M \models_{\Sigma} \varphi \}. \quad (2.1)$$

Analogously, we define for a given subcategory $M \subseteq \text{Mod}(\Sigma)$ of $\Sigma$-models, the set of theorems $\text{th}(\Sigma)(M) \subseteq \text{Sen}(\Sigma)$ given by those sentences $\varphi \in \text{Sen}(\Sigma)$ which are satisfied by all models in $M$, i.e., we have
For any functor $F : C \to D$ the existential image powerset functor $f : P(A) \to P(B)$, by setting $f(A') = \{ f(a) | a \in A' \}$ for any $A' \in \mathcal{P}(A)$, and $\mathcal{P}(D) \to \mathcal{P}(C)$, where for any $M \subseteq D$: $C \in |F^{-1}(M)|$ iff $F(C) \in M$ and $g$ an arrow in $F^{-1}(M)$ iff $F(g)$ an arrow in $M$.}

Thus, for any function $f : A \to B$ the existential image powerset functor $f : P(A) \to P(B)$, by setting $f(A') = \{ f(a) | a \in A' \}$ for any $A' \in \mathcal{P}(A)$, and $\mathcal{P}(D) \to \mathcal{P}(C)$, where for any $M \subseteq D$: $C \in |F^{-1}(M)|$ iff $F(C) \in M$ and $g$ an arrow in $F^{-1}(M)$ iff $F(g)$ an arrow in $M$.

In such a way, we obtain for any signature morphism $\phi : \Sigma_1 \to \Sigma_2$ functors

\[ \text{Sen}(\phi) : \text{Spec}(\Sigma_1) \to \text{Spec}(\Sigma_2) \quad \text{and} \quad \text{Mod}(\phi)^{-1} : \text{Sub}(\Sigma_1) \to \text{Sub}(\Sigma_2). \]

Thus the presented simple abstraction step from the conceptual level of single sentences and single models to the level of specifications and subcategories of models

\[ th(\Sigma)(M) \overset{\text{def}}{=} \{ \varphi \in \text{Sen}(\Sigma) | \forall M \in |M| : M \models \Sigma \varphi \}. \]
produces a diagram of adjunctions that keeps the complete information provided by the triple \((\text{Sen}, \text{Mod}, \models)\) for any signature morphism \(\phi : \Sigma_1 \to \Sigma_2\):

\[
\begin{array}{ccc}
\Sigma_1 & \xrightarrow{\text{th}(\Sigma_1)} & \text{Spec}(\Sigma_1) \\
\downarrow \phi \quad \text{Mod}(\phi)^{-1} & \\n\Sigma_2 & \xrightarrow{\text{th}(\Sigma_2)} & \text{Spec}(\Sigma_2)
\end{array}
\]

\[
\begin{array}{ccc}
\text{Sub}(\Sigma_1) & \xrightarrow{\text{mod}(\Sigma_1)} & \text{Spec}(\Sigma_1) \\
\downarrow \phi \quad \text{Mod}(\phi)^{-1} & \\n\text{Sub}(\Sigma_2) & \xrightarrow{\text{mod}(\Sigma_2)} & \text{Spec}(\Sigma_2)
\end{array}
\]

The crucial insight is now that the institution condition turns out to be equivalent to a commutativity requirement for this diagram:

**Lemma 2.16 (Institution Condition)**

Let \(I = (\text{Sign}, \text{Sen}, \text{Mod}, \models)\) be an institution. Then we have for any signature morphism \(\phi : \Sigma_1 \to \Sigma_2\) in \(\text{Sign}\):

1. The following inequality is equivalent to condition \(\text{IC}_{\text{Sen} \Rightarrow \text{Mod}}\):
   
   \[
   \text{th}(\Sigma_2) (\text{Sen}(\phi)(\Gamma_1)) \subseteq \text{Mod}(\phi)^{-1} (\text{mod}(\Sigma_1)(\Gamma_1)) \quad \text{for all } \Gamma_1 \in |\text{Spec}(\Sigma_1)|.
   \]

Any of the following inequalities is equivalent to condition \(\text{IC}_{\text{Mod} \Rightarrow \text{Sen}}\):

2. \(\text{Mod}(\phi)^{-1} (\text{mod}(\Sigma_1)(\Gamma_1)) \subseteq \text{mod}(\Sigma_2) (\text{Sen}(\phi)(\Gamma_1))\), for all \(\Gamma_1 \in |\text{Spec}(\Sigma_1)|\).
3. \(\text{th}(\Sigma_2) (\text{Mod}(\phi)^{-1}(\Sigma_1)) \supseteq \text{Sen}(\phi) (\text{th}(\Sigma_1)(\Sigma_1))\), for all \(\Sigma_1 \in |\text{Sub}(\Sigma_1)|\).

**Proof.** \(\text{IC}_{\text{Sen} \Rightarrow \text{Mod}}\) implies inequality 1: For any \(M_2 \in |\text{Mod}(\Sigma_2)|\) we obtain

\[
\begin{align*}
M_2 &\in |\text{mod}(\Sigma_2)(\text{Sen}(\phi)(\Gamma_1))| \\
\Leftrightarrow M_2 &\models_{\Sigma_2} \text{Sen}(\phi)(\varphi_1) \quad \text{for all } \varphi_1 \in \Gamma_1 \quad (\text{def. } \text{mod}(\Sigma_2)) \\
\Rightarrow M_2 &\models_{\Sigma_1} \text{Mod}(\phi)(M_2) \models_{\Sigma_1} \varphi_1 \quad \text{for all } \varphi_1 \in \Gamma_1 \quad (\text{IC}_{\text{Sen} \Rightarrow \text{Mod}}) \\
\Leftrightarrow M_2 &\in |\text{Mod}(\phi)^{-1}(\text{mod}(\Sigma_1)(\Gamma_1))| \quad (\text{def. } \text{mod}(\Sigma_1)) \\
\Leftrightarrow M_2 &\in |\text{Mod}(\phi)^{-1}(\text{mod}(\Sigma_1)(\Gamma_1))| \quad (\text{def. } \text{Mod}(\phi)^{-1})
\end{align*}
\]

Inequality 1 implies \(\text{IC}_{\text{Sen} \Rightarrow \text{Mod}}\): For any \(M_2 \in |\text{Mod}(\Sigma_2)|\) we obtain

\[
\begin{align*}
M_2 &\models_{\Sigma_2} \text{Sen}(\phi)(\varphi_1) \iff M_2 &\in |\text{mod}(\Sigma_2)(\text{Sen}(\phi)(\varphi_1))| \quad (\text{def. } \text{mod}(\Sigma_2)) \\
\Rightarrow M_2 &\in |\text{Mod}(\phi)^{-1}(\text{mod}(\Sigma_1)(\varphi_1))| \quad (\text{inequality 1}) \\
\Leftrightarrow M_2 &\models_{\Sigma_1} \text{Mod}(\phi)(M_2) \models_{\Sigma_1} \varphi_1 \quad (\text{def. } \text{Mod}(\phi)^{-1}) \\
\Leftrightarrow M_2 &\models_{\Sigma_1} \text{Mod}(\phi)(M_2) \models_{\Sigma_1} \varphi_1 \quad (\text{def. } \text{mod}(\Sigma_1))
\end{align*}
\]

The equivalence of \(\text{IC}_{\text{Mod} \Rightarrow \text{Sen}}\) and inequality 2 is shown analogously.
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Inequality 2 implies inequality 3: Due to the unit law in Proposition 2.14, the monotonicity of \( \text{Mod}(\phi)^{-1} \), and inequality 2 we obtain

\[
\text{Mod}(\phi)^{-1}(M_1) \subseteq \text{Mod}(\phi)^{-1}(\text{mod}(\Sigma_1)(\text{th}(\Sigma_1)(M_1))) 
\subseteq \text{mod}(\Sigma_2)(\text{Sen}(\phi)(\text{th}(\Sigma_1)(M_1)))
\]

but, by the adjointness law in Proposition 2.14 this is equivalent to

\[
\text{th}(\Sigma_2)(\text{Mod}(\phi)^{-1}(M_1)) \supseteq \text{Sen}(\phi)(\text{th}(\Sigma_1)(M_1))
\]

Inequality 3 implies inequality 2: The instance of inequality 3

\[
\text{th}(\Sigma_2)(\text{Mod}(\phi)^{-1}(\text{mod}(\Sigma_1)(\Gamma_1))) \supseteq \text{Sen}(\phi)(\text{th}(\Sigma_1)(\text{mod}(\Sigma_1)(\Gamma_1)))
\]

is due to the adjointness law in in Proposition 2.14 equivalent to

\[
\text{Mod}(\phi)^{-1}(\text{mod}(\Sigma_1)(\Gamma_1)) \subseteq \text{mod}(\Sigma_2)(\text{Sen}(\phi)(\text{th}(\Sigma_1)(\text{mod}(\Sigma_1)(\Gamma_1))))
\]

But, the counit law in Proposition 2.14, the monotonicity of \( \text{Sen}(\phi) \) and the functor property of \( \text{mod}(\Sigma_2) \) entail

\[
\text{mod}(\Sigma_2)(\text{Sen}(\phi)(\text{th}(\Sigma_1)(\text{mod}(\Sigma_1)(\Gamma_1)))) \subseteq \text{mod}(\Sigma_2)(\text{Sen}(\phi)(\Gamma_1))
\]

Concerning Lemma 2.16.1 and 2.16.2, the reader has to bear in mind that the inverse image of a full subcategory becomes a full subcategory as well.

Since the categories \( \text{Spec}(\Sigma) \) and \( \text{Sub}(\Sigma) \) are partial order categories, i.e., only the existence of arrows matters, lemma 2.16 shows that the institution condition is equivalent to the following equation:

\[
\text{mod}(\Sigma_1); \text{Mod}(\phi)^{-1} = \text{Sen}(\phi); \text{mod}(\Sigma_2).
\] (2.3)

To summarize our analysis and to give an overall picture we have to remind:

**FACT 2.17** (Power Construction Functors)
1. The assignments \( A \mapsto \mathcal{P}(A) \) and \( (f : A \to B) \mapsto (f : \mathcal{P}(A) \to \mathcal{P}(B)) \), due to Fact 2.15, define a functor \( \mathcal{P} : \text{Set} \to \text{Cat} \).
2. The assignments \( C \mapsto \mathcal{P}(C) \) and \( (F : C \to D) \mapsto (F^{-1} : \mathcal{P}(D) \to \mathcal{P}(C)) \), due to Fact 2.15, define a functor \( \mathcal{P}^{-1} : \text{Cat}^{op} \to \text{Cat} \).

Now, our definitions can be compressed to

\[
\text{Spec} \overset{\text{def}}{=} \text{Sen}; \mathcal{P} : \text{Sign} \to \text{Cat} \quad \text{and} \quad \text{Sub} \overset{\text{def}}{=} \text{Mod}^{op}; \mathcal{P}^{-1} : \text{Sign} \to \text{Cat}
\] (2.4)

and equation 2.3 says that \( \text{mod}(\Sigma) \) is the component at \( \Sigma \) of an indexed functor \( \text{mod} : \text{Spec} \Rightarrow \text{Sub} : \text{Sign} \to \text{Cat} \), i.e., the institution condition is equivalent to a naturality requirement.

**THEOREM 2.18** (Indexed Institutional Frame)
A 4-tuple \( \mathcal{I} = (\text{Sign}, \text{Sen}, \text{Mod}, \models) \) is an Institutions iff the 5-tuple \( \mathcal{P}(\mathcal{I}) = (\text{Sign}, \text{Spec}, \text{Sub}, \text{mod}, \text{th}) \) is an indexed frame, called the **INDEXED INSTITUTIONAL FRAME** for \( \mathcal{I} \).
Proof. Immediately by Proposition 2.14, Lemma 2.16, and Fact 2.17.

Similar to the institution condition also the comorphism condition turns out to be equivalent to a commutativity requirement:

Lemma 2.19 (Comorphism Condition)
Let $\mu = (\Phi, \alpha, \beta) : I \to I'$ be an institution comorphism. Then for any signature $\Sigma \in |\text{Sign}|$ the comorphism condition is equivalent to the equation

$$\text{mod}(\Sigma); \beta(\Sigma)^{-1} = \alpha(\Sigma); \text{mod}(\Phi(\Sigma)).$$

Proof. Analogously to the proof of Lemma 2.19.

In such a way, composing the components of an institution comorphism with the power construction functors, we obtain two natural transformations

$$\alpha \cdot \mathcal{P} : \text{Spec} \Rightarrow \Phi; \text{Spec}$$
and

$$\beta^{op} \cdot \mathcal{P} : \text{Sub} \Rightarrow \Phi; \text{Sub}^{op},$$
where we have to bear in mind that opposing categories causes an inversion for natural transformations, i.e., $\beta^{op} : \text{Mod}^{op} \Rightarrow \Phi^{op}; \text{Mod}^{op} : \text{Sign} \to \text{Cat}^{op}$. Lemma 2.19 can be reformulated now according to Theorem 2.18 as

Theorem 2.20 (Map of Indexed Institutional Frame)
A triple $\mu = (\Phi, \alpha, \beta) \in \text{InstCom}$ is an institution comorphism from $I$ into $I'$ iff the triple $P(\mu) \defeq (\Phi, \alpha \cdot P, \beta^{op} \cdot P^{-})$ is a map of indexed frames from the indexed institutional frame $P(I)$ into the indexed institutional frame $P(I')$.

Since the transitions from $I$ to $P(I)$ and from $\mu$ to $P(\mu)$, respectively, are defined in a well-structured way by post-composing the corresponding components with power construction functors we obtain

Proposition 2.21 (Map of Indexed Institutional Frame)
The assignments $I \mapsto P(I)$ due to Theorem 2.18 and $\mu \mapsto P(\mu)$, due to Theorem 2.20 define a functor $P : \text{InstCom} \to \text{IndFra}$.

Proof. Identities:

$$P(id_I) = (id_{\text{Sign}}, id_{\text{Sen}}, P, id_{\text{Mod}}; P^{op}, P^{-})$$ (Def. 2.12 and Theorem 2.20)

$$= (id_{\text{Sign}}, id_{\text{Sen}}, P, id_{\text{Mod}}; P^{op}, P^{-})$$ (Equation 1.2)

$$= (id_{\text{Sign}}, id_{\text{Spec}}, id_{\text{Sub}})$$ (Equation 2.4)

$$= id_{P(I)}$$ (Definition 2.5)

Composition:

$$P(\mu); P(\mu')$$

$$= (\Phi, \alpha \cdot P, \beta^{op} \cdot P^{-}); (\Phi', \alpha' \cdot P, \beta'^{op} \cdot P^{-})$$ (Theorem 2.20)

$$= (\Phi, \Phi' \cdot (\alpha \cdot P); (\Phi', \alpha' \cdot P)); (\beta^{op} \cdot P^{-}); ((\Phi, \alpha \cdot P) \cdot (\beta^{op} \cdot P^{-})$$ (Equation 1.4)

$$= (\Phi, \Phi' \cdot (\alpha; (\Phi, \alpha' \cdot P)); (\beta^{op}; (\Phi, \beta'^{op}) \cdot P^{-})$$ (opposing)

$$= P(\Phi; \Phi', \alpha; (\Phi, \alpha' \cdot P), (\Phi^{op}, \beta'; \beta^{op}) \cdot P^{-})$$ (Theorem 2.20)

$$= P(\mu; \mu')$$ (Definition 2.12)
Remark 2.22 (General Map of Institutions)
Note, that the functor $P : \text{InstCom} \to \text{IndFra}$ is not full, and that the original general concept of a “map of institution” in [20] covers exactly also all the maps between the corresponding indexed institutional frames not reached by this functor. In [12] those general maps are classified as “theoroidal comorphisms”.

3 Fibred Logical Systems

3.1 Split Opfibrations

In generalizing the formation of the disjoint union of a family of sets all the components of an indexed category can be put together into a single “flat” category by the so-called Grothendieck construction. Note that the following presentation differs with respect to the more traditional one given in the literature [3].

Definition 3.1 (Grothendieck Constructions and Indexed Categories)
Given an indexed category $C : \text{Ind} \to \text{Cat}$ we define the category $\text{Flat}(C)$ as follows:

- **objects:** are pairs $\langle i, a \rangle$ where $i \in |\text{Ind}|$ and $a \in |C(i)|$.
- **arrows:** from $\langle i, a \rangle$ to $\langle j, b \rangle$ are pairs $\langle \sigma, f \rangle$ where $\sigma : i \to j$ is an arrow in $\text{Ind}$ and $f : b \to C(\sigma)(a)$ is an arrow in $C(j)$.
- **composition:** Given arrows $\langle \sigma, f \rangle : \langle i, a \rangle \to \langle j, b \rangle$ and $\langle \tau, g \rangle : \langle j, b \rangle \to \langle k, c \rangle$ in $\text{Flat}(C)$, let $\langle \sigma, f \rangle ; \langle \tau, g \rangle \overset{\text{def}}{=} \langle \sigma ; \tau, g ; C(\tau)(f) \rangle$.
- **identities:** Given an object $\langle i, a \rangle$ of $\text{Flat}(C)$, its identity is defined as $\text{id}_{\langle i, a \rangle} \overset{\text{def}}{=} \langle \text{id}_i, \text{id}_a \rangle : \langle i, a \rangle \to \langle i, a \rangle$.
- Moreover we obtain a projection functor $\text{Pr}_C : \text{Flat}(C) \to \text{Ind}$ with $\text{Pr}_C(\langle i, a \rangle) = i$ and $\text{Pr}_C(\langle \sigma, f \rangle) = \sigma$ for any $\langle i, a \rangle$ and any $\langle \sigma, f \rangle : c \to C(\tau)(C(\sigma)(a))$ in $\text{Flat}(C)$.

$$
\begin{array}{ccc}
  & & c \\
  & s \downarrow & \\
  b & \downarrow f & C(\tau)(b) \\
  \downarrow & & \downarrow C(\tau)(f) \\
a & C(\sigma)(a) & C(\tau)(C(\sigma)(a)) \\
\end{array}
$$

$$
\begin{array}{ccc}
C(i) & \overset{C(\sigma)}{\longrightarrow} & C(j) \\
\downarrow & & \downarrow C(\tau) \\
\overset{C(\tau)}{\longrightarrow} & C(k) \\
\end{array}
$$

Example 3.2
Given an Institution I, we consider first the indexed category $\text{Spec} : \text{Sign} \to \text{Cat}$. Then we have $\text{Flat}(\text{Spec})$ as follows: the objects are pairs $\langle \Sigma, \Gamma \rangle$, where $\Sigma \in |\text{Sign}|$.

$$
\begin{array}{ccc}
i & \overset{\sigma}{\longrightarrow} & j \\
\downarrow & & \downarrow \tau \\
& \overset{\tau}{\longrightarrow} & k \\
\end{array}
$$
and \( \Gamma \) is an object of \( Spec(\Sigma) \). Now an arrow \( \langle \phi, \Sigma \rangle : \langle \Sigma_1, \Gamma_1 \rangle \rightarrow \langle \Sigma_2, \Gamma_2 \rangle \) is such that \( \sigma : \Sigma_1 \rightarrow \Sigma_2 \) is an arrow in \( Sign \) and \( \Gamma_2 \supseteq Sen(\phi)(\Gamma_1) \) is a specification morphism in \( Spec(\Sigma_2) \). There is also the projection functor \( P_{Spec} : Flat(Spec) \rightarrow \text{Sign} \), such that \( \langle \Sigma, \Gamma \rangle \mapsto \Sigma \). By the same token, considering \( Sub : \text{Sign} \rightarrow \text{Cat} \) an object in \( Flat(Sub) \) is a pair \( \langle \Sigma, M \rangle \) such that \( \Sigma \) is an object of \( \text{Sign} \) and \( M \) is a category of models and an object of \( Sub(\Sigma) \). An arrow \( \langle \phi, \subseteq \rangle : \langle \Sigma_1, M_1 \rangle \rightarrow \langle \Sigma_2, M_2 \rangle \) is such that \( \phi : \Sigma_1 \rightarrow \Sigma_2 \) is a signature morphism and \( M_2 \subseteq Mod(\phi)^{-1}(M_1) \) is a model morphism in \( Sub(\Sigma_2) \). Note that we also have a projection functor \( P_{Sub} : Flat(Sub) \rightarrow \text{Sign} \).

One question that arises naturally is that if there is an axiomatic characterization of those “fibred categories” resulting from the Grothendieck construction. First, we look for an axiomatization of the fact that an object \( b \) can be seen as the result of applying to an object \( a \) a translation induced by a morphism \( \sigma \) between indices.

**Definition 3.3 (Op-Cartesian Arrows)**

Let \( \text{Ind} \) and \( \mathcal{C} \) be arbitrary categories and \( P : \mathcal{C} \rightarrow \text{Ind} \) be a functor. Let \( \sigma : i \rightarrow j \) be an arrow in \( \mathcal{C} \) and \( a \) be an object of \( \mathcal{C} \) such that \( P(a) = i \). An arrow \( k : a \rightarrow b \) in \( \mathcal{C} \) is \textit{op-cartesian} for \( \sigma \) and \( a \) if the following two conditions are satisfied:

**OC-1** \( P(k) = \sigma \);

**OC-2** given any other arrow \( v : a \rightarrow c \), each factorization of \( P(v) \) through \( \sigma \) uniquely determines a factorization of \( v \) through \( k \), i.e., for any arrow \( \epsilon : j \rightarrow P(c) \) in \( \mathcal{C} \) such that \( P(v) = \sigma ; \epsilon \) there exists a unique \( w : b \rightarrow c \) such that \( v = k ; w \) and \( P(w) = \epsilon \).

![Diagram](image_url)

**Definition 3.4 (Opfibration)**

A functor \( P : \mathcal{C} \rightarrow \text{Ind} \) is said to be an \textit{opfibration} if there exists an op-cartesian arrow for every arrow \( \sigma : i \rightarrow j \) and object \( a \in [\mathcal{C}] \) for which \( P(a) = i \) in \( \text{Ind} \). If \( P : \mathcal{C} \rightarrow \text{Ind} \) is an opfibration, then we say that \( \mathcal{C} \) is \textit{fibered over} \( \text{Ind} \).

We can now make for any object \( a \) and any morphism \( \sigma \) an “individual local choice” between all the possible results of applying to \( a \) the translation induced by \( \sigma \).

**Definition 3.5 (Cleavage)**

If \( P : \mathcal{C} \rightarrow \text{Ind} \) is an opfibration, a \textit{cleavage} for \( P : \mathcal{C} \rightarrow \text{Ind} \) is a function \( \kappa \) that takes an arrow \( \sigma : i \rightarrow j \) and \( a \) such that \( P(a) = i \) to an arrow \( \kappa(\sigma, a) \) that is op-cartesian for \( \sigma \) and \( a \).
The individual local choices will, in general, not ensure global compositionality, i.e., $\kappa(\sigma, a); \kappa(\tau, b)$ will be, in general, different from $\kappa(\sigma; \tau, a)$. But, in some cases we can globally coordinate the local choices in such a way that we obtain

**Definition 3.6 (Split Opfibrations)**

Let $P : C \to \text{Ind}$ be an opfibration. Then, a given cleavage $\kappa$ is called **split** if the following conditions are satisfied:

**SC-1** $\kappa(id_i, a) = id_a$, where $P(a) = i$.

**SC-2** $\kappa(\sigma, a); \kappa(\tau, b) = \kappa(\sigma; \tau, a)$, where $\sigma : i \to j, \tau : j \to k$, such that $P(a) = i$ and $P(b) = j$.

![Diagram]

An opfibration with a **splitting** $\kappa$ is called a **split opfibration**.

Note that the universal property of op-cartesian arrows provides for arbitrary fibrations at least an isomorphism $i_{\sigma, \tau}$ such that $\kappa(\sigma, a); \kappa(\tau, b); i_{\sigma, \tau} = \kappa(\sigma; \tau, a)$.

For indexed categories the global coordination of the local choices is “built-in” thus the property “split obfibration” has to be part of our intended axiomatization.

**Fact 3.7 (Grothendieck Construction and Split Obfibrations)**

Given an indexed category $C : \text{Ind} \to \text{Cat}$, $\text{Flat}(C)$ is a category and the projection functor $Pr_C : \text{Flat}(C) \to \text{Ind}$, defined in 3.1, is a **split opfibration**, with splitting

$$\kappa(\sigma, \langle i, a \rangle) = \langle \sigma, id_{C(\sigma)(a)} \rangle : \langle i, a \rangle \to \langle j, C(\sigma)(a) \rangle$$

for any arrow $\sigma : i \to j$ of $C$ and object $\langle i, a \rangle$ of $\text{Flat}(C)$.

**Proof.** Note that $\text{Flat}(C)$ is a category by definition 3.1. We must show that $\kappa(\sigma, \langle i, a \rangle)$ is op-cartesian for any $\sigma : i \to j$ and and any $\langle i, a \rangle$. We consider an arbitrary arrow $\langle \delta, h \rangle : \langle i, a \rangle \to \langle k, c \rangle$ such that $\sigma; \epsilon = \delta$ for a certain arrow $\epsilon : j \to k$ and obtain the following diagram

![Diagram]
We have to show that there is a unique \( \omega : \langle j, C(\sigma)(a) \rangle \to \langle k, c \rangle \) such that \( \langle \delta, h \rangle = \langle \sigma, id_{C(\sigma)(a)} \rangle ; \omega \) and \( Pr_C(\omega) = \epsilon \). The second condition implies \( \omega = \langle \epsilon, x \rangle \) thus we obtain

\[
\langle \sigma, id_{C(\sigma)(a)} \rangle ; \omega = \langle \sigma, id_{C(\sigma)(a)} \rangle ; \langle \epsilon, x \rangle
\]

(\text{SECOND CONDITION})

\[
= \langle \sigma; \epsilon, x; C(\epsilon)(id_{C(\sigma)(a)}) \rangle
\]

(\text{COMPOSITION IN Flat}(C))

\[
= \langle \sigma; \epsilon, x; id_{C(\epsilon)(C(\sigma)(a))} \rangle
\]

(\text{FUNCTOR PROPERTY})

\[
= \langle \delta, x \rangle
\]

(\text{ASSUMPTION AND IDENTITY LAW})

In such a way, the first condition entails uniquely \( x = h \) thus we obtain finally the required unique \( \omega = \langle \epsilon, h \rangle \).

It remains to show that the splitting preserves identities and composition. Condition SC-1 is ensured by \( \kappa(\epsilon, \langle i, a \rangle) = \langle id_{\epsilon}, \langle i, a \rangle \rangle = \langle id_{\epsilon}, id_{C(\sigma)(a)} \rangle = \langle id_{\epsilon}, id_{\langle i, a \rangle} \rangle \) as a particular case of Fact 3.7. The first equality is due to definition of \( \kappa \). The second by the functor property, the third by the identity functor and the fourth by definition of identities in Flat(\( C \)).

On the other hand, condition SC-2 is obtained by

\[
\kappa(\sigma; \epsilon, \langle i, a \rangle)
\]

(DEFINITION OF \( \kappa \))

\[
= \langle \sigma; \epsilon, id_{C(\sigma)(\epsilon)(a)} \rangle
\]

(FUNCTOR PROPERTY)

\[
= \langle \sigma; \epsilon, id_{C(\sigma)(\epsilon)(\langle i, a \rangle)} \rangle
\]

(APPLICATION)

\[
= \langle \sigma; \epsilon, C(\epsilon)(id_{C(\sigma)(\epsilon)(a)}) \rangle
\]

(FUNCTOR PROPERTY)

\[
= \langle \sigma; \epsilon, id_{C(\sigma)(\epsilon)(\langle i, a \rangle)} ; C(\epsilon)(id_{C(\sigma)(\epsilon)}(a)) \rangle
\]

(DEFINITION OF \( \kappa \))

As a special case of Fact 3.7 we obtain that any arrow \( \langle \sigma, f \rangle : \langle i, a \rangle \to \langle j, c \rangle \) can be factored through the splitting as \( \langle \sigma, id_{C(\sigma)(a)} ; (id_{j}, f) \rangle \). This factorization reflects perfectly the fact that the arrows in Flat(\( C \)) are defined as pairs of arrows.

\textbf{Example 3.8}

Consider the opfibration \( Pr_{Spec} : Flat(\text{Spec}) \to \text{Sign} \). Now for any \( \phi : \Sigma_1 \to \Sigma_2 \) in
Let and any specification \( \langle \Sigma_1, \Gamma_1 \rangle \), the splitting is given by

\[
\kappa_{\text{Spec}}(\langle \phi, (\Sigma_1, \Gamma_1) \rangle) = \langle \phi, \text{id}_{\text{Sen}(\phi)(\Gamma_1)} \rangle : \langle \Sigma_1, \Gamma_1 \rangle \rightarrow \langle \Sigma_2, \text{Sen}(\phi)(\Gamma_1) \rangle.
\]

Note that the opcartesian arrow above shows how to typecast an \( \Sigma_1 \)-specification into an \( \Sigma_2 \)-one. Also any specification morphism \( \langle \phi, \mathcal{S} \rangle : \langle \Sigma_1, \Gamma_1 \rangle \rightarrow \langle \Sigma_2, \Gamma_2 \rangle \) can be factored as \( \langle \phi, \text{id}_{\text{Sen}(\phi)(\Gamma_1)} \rangle; \langle \mathcal{S}, \mathcal{S} \rangle \) : \( \langle \Sigma_2, \text{Sen}(\phi)(\Gamma_1) \rangle \rightarrow \langle \Sigma_2, \Gamma_2 \rangle \).

Consider now the opfibration \( \mathcal{F}_{\text{Sub}} : \text{Flat(Sub)} \rightarrow \text{Sign} \). Given any \( \phi : \Sigma_1 \rightarrow \Sigma_2 \) in \( \text{Sign} \) and any \( \langle \Sigma_1, M_1 \rangle \), the splitting is given by

\[
\kappa_{\text{Sub}}(\langle \phi, (\Sigma_1, M_1) \rangle) = \langle \phi, \text{id}_{\text{Mod}(\phi)^{-1}(M_1)} \rangle : \langle \Sigma_1, M_1 \rangle \rightarrow \langle \Sigma_2, \text{Mod}(\phi)^{-1}(M_1) \rangle.
\]

Note that \( \text{Mod}(\phi)^{-1} \) is not only reindexing, but also taking all \( \Sigma_2 \) extensions of the models in the category \( M_1 \). Again, any arrow \( \langle \phi, \mathcal{S} \rangle : \langle \Sigma_1, M_1 \rangle \rightarrow \langle \Sigma_2, M_2 \rangle \) can be factored as \( \langle \phi, \text{id}_{\text{Mod}(\phi)^{-1}(M_1)} \rangle; \langle \mathcal{S}, \mathcal{S} \rangle \), where \( \langle \text{id}_{\text{Mod}(\phi)^{-1}(M_1)}, \mathcal{S} \rangle : \langle \Sigma_2, \text{Mod}(\phi)^{-1}(M_1) \rangle \rightarrow \langle \Sigma_2, M_2 \rangle \).

Inversing the Grothendieck construction we can split up a “fibred category” into disjoint components.

**Definition 3.9 (Fibers)**

For any functor \( P : C \rightarrow \text{Ind} \), the fiber \( P_F(i) \) over an object \( i \) of \( \text{Ind} \) is the subcategory of \( C \) given by the collection of objects \( a \) such that \( P(a) = i \), and the arrows \( u \) for which \( P(u) = \text{id}_i \), i.e., \( P_F(i) \defeq P^{-1}(\text{id}_i) \).

**Example 3.10**

Consider the functor \( \mathcal{F}_{\text{Spec}} : \text{Flat(Sub)} \rightarrow \text{Sign} \). Then for each object \( \Sigma \) of \( \text{Sign} \) we may define (recover) the set of sentences of \( \Sigma \) as typed specifications in \( \text{Spec}_{\mathcal{F}}(\Sigma) \defeq \mathcal{F}_{\text{Sub}}^{-1}(\Sigma) \). In the same way, considering the functor \( \mathcal{F}_{\text{Sub}} : \text{Flat(Sub)} \rightarrow \text{Sign} \) we recover the category of models of the signature \( \Sigma \) as typed models in \( \text{Sub}_{\mathcal{F}}(\Sigma) \defeq \mathcal{F}_{\text{Sub}}^{-1}(\Sigma) \). Note that the objects of \( \text{Spec}_{\mathcal{F}}(\Sigma) \) are pairs \( (\Sigma, \Gamma) \), where \( \Gamma \subseteq \text{Sen}(\Sigma) \). Similarly, the objects of \( \text{Sub}_{\mathcal{F}}(\Sigma) \) are pairs \( (\Sigma, M) \), where \( M \subseteq \text{Mod}(\Sigma) \).

The universal property of op-cartesian arrows ensures that any individual local choice provides “local functoriality”.

**Definition 3.11 (Cleavages induce Functors)**

Let \( P : C \rightarrow \text{Ind} \) be an opfibration, with cleavage \( \kappa \). Then we define:

1. Let \( P_F(i) \) be the fiber over \( i \) for each object \( i \) of \( \text{Ind} \).
2. For \( \sigma : i \rightarrow j \) in \( \text{Ind} \) and \( a \) an object of \( P_F(i) \), \( P_F(\sigma)(a) \) is defined to be the codomain of the (op-cartesian) arrow \( \kappa(\sigma, a) \).
3. For \( \sigma : i \rightarrow j \) in \( \text{Ind} \) and \( u : a \rightarrow a' \) in \( P_F(i) \), \( P_F(\sigma)(u) \) is the unique arrow from \( P_F(\sigma)(a) \) to \( P_F(\sigma)(a') \) (by universality - condition OA-2 - of \( \kappa(\sigma, a) \)) for which

\[
\kappa(\sigma, a); P_F(\sigma)(u) = u; \kappa(\sigma, a')
\]
Fact 3.12
Let \( P : C \to \text{Ind} \) be an opfibration and \( \sigma : i \to j \) an arrow in \( \text{Ind} \). Then, definition 3.11 defines a functor \( FP(\sigma) : FP(i) \to FP(j) \).

**Proof.** First note that the map \( FP(\sigma) : FP(i) \to FP(j) \) is well-defined by condition 3 in definition 3.11 above. It remains to show preservation of composition and identities.

**Preservation of composition:** Let \( u : a \to a' \) and \( v : a' \to a'' \) be arrows in \( F(i) \) and consider the diagram below:

By universality of \( \kappa(\sigma,a) \), \( FP(\sigma)(u) \) is the unique arrow that makes the top bottom diagram commutes. Also, by universality of \( \kappa(\sigma,a') \), \( FP(\sigma)(v) \) is the unique arrow that makes the top front diagram commutes. So \( FP(\sigma)(u); FP(\sigma)(v) \) is a solution to the equation

\[(u;v);\kappa(\sigma,a'') = \kappa(\sigma,a); (?) \]

However, by definition 3.11, \( FP(\sigma)(u;v) \) also solves the equation, and therefore they must be equal.

**Preservation of identities:** Let \( id_a : a \to a \) be an arrow in \( F_P(i) \) and consider the diagram below:
By definition 3.11, $F_P(\sigma)(id_a)$ is the unique arrow that makes this diagram commute. Since $id_b = id_{F_P(\sigma)(a)}$ also does the job, they are equal.

Example 3.13 (Specification and Inverse Image Model Functors)
Considering the functor $Pr_{Spec} : Flat(Spec) \to \text{Ind}$, then each $\phi : \Sigma_1 \to \Sigma_2$ induces a functor $F_{Pr_{Spec}} : Pr_{Spec}^{-1}(id_{\Sigma_1}) \to Pr_{Spec}^{-1}(id_{\Sigma_2})$, written $Spec_{Pr}(\phi) : Spec_{Pr}(\Sigma_1) \to Spec_{Pr}(\Sigma_2)$ (see example 3.10). Note that $Spec_{Pr}(\phi)$ is not equal, but actually natural equivalent to $Spec(\phi)$.

The following diagram illustrates the construction of this map:

In the same way, considering the opfibration $Pr_{Sub} : Flat(Sub) \to \text{Sign}$, the previous fact says that each arrow $\phi : \Sigma_1 \to \Sigma_2$ induces a functor $F_{Pr_{Sub}} : Pr_{Sub}^{-1}(id_{\Sigma_1}) \to Pr_{Sub}^{-1}(id_{\Sigma_2})$, written as $Sub_{Pr}(\phi) : Sub_{Pr}(\Sigma_1) \to Sub_{Pr}(\Sigma_2)$ (see again example 3.10). Note that $Sub_{Pr}(\phi)$ is actually natural equivalent to $Sub(\phi) = Mod(\phi)^{-1}$.

A global coordination of the individual local choices ensures also global functoriality or, to put it the other way around, the property “split obfibration” is exactly the axiomatization we have been looking for.

Fact 3.14
Let $P : C \to \text{Ind}$ be a split opfibration. Then, definition 3.11 defines an indexed category $F_P : \text{Ind} \to \text{Cat}$.
Proof. Note that $F_P : \text{Ind} \rightarrow \text{Cat}$ is a map by fact 3.12. To verify that this map is a functor we need to show preservation of composition and identities.

Preservation of composition: Let $\sigma : i \rightarrow j$, $\tau : j \rightarrow k$ be arrows in $\text{Ind}$, $u : a \rightarrow a'$ an arrow in $F_P(i)$, and consider the diagram below:

$$
\begin{array}{c}
da & \xrightarrow{\kappa(\sigma, a)} & c \\
\downarrow{u} & & \downarrow{F_P(\sigma; \tau)(u)} \\
a' & \xrightarrow{\kappa(\sigma; \tau, a')} & c'
\end{array}
$$

$F_P(\sigma; \tau)(u)$ is the unique arrow such that $\kappa(\sigma; \tau, a); F(\sigma; \tau)(u) = u; \kappa(\sigma; \tau, a')$ by universality of $\kappa(\sigma; \tau, a)$. Now, since $\kappa$ is splitting, this is equivalent of assuming the commutativity of the top squares in the diagram below:

$$
\begin{array}{c}
a & \xrightarrow{\kappa(\sigma, a)} & b \\
\downarrow{u} & & \downarrow{F_P(\sigma)(u)} \\
b' & \xrightarrow{\kappa(\sigma', \tau)} & c' \\
\downarrow{\kappa(\tau, \tau')} & & \downarrow{\kappa(\tau, \tau')} \\
c & \xrightarrow{\kappa(\tau, \tau')} & c'
\end{array}
$$

But now, both $[F_P(\sigma); F_P(\tau)](u)$ and $F_P(\sigma; \tau)(u)$ are solutions to the equation $\kappa(\sigma; \tau, a)(?) = u; \kappa(\sigma; \tau, a')$. Since this solution must be unique, they are equal.

Preservation of identities: Let $u : a \rightarrow a'$ be an arrow in $F_P(i)$ and consider the diagram below:

$$
\begin{array}{c}
a & \xrightarrow{\kappa(\sigma, a)} & b \\
\downarrow{u} & & \downarrow{F_P(\sigma)(u)} \\
b' & \xrightarrow{\kappa(\sigma', \tau)} & c' \\
\downarrow{\kappa(\tau, \tau')} & & \downarrow{\kappa(\tau, \tau')} \\
c & \xrightarrow{\kappa(\tau, \tau')} & c'
\end{array}
$$
By universality of \( \kappa(id_i, a) \), \( F_P(id_i)(u) \) is the unique arrow such that the top square commutes. Moreover, since \( \kappa \) is splitting we have that \( \kappa(id_i, a) = id_a \) and \( \kappa(id_i, a') = id_{a'} \). This means that also \( u = id_{F_P(i)(u)} \) makes the top square commute. Since the solution of the equation \( u; \kappa(id_i, a') = \kappa(id_i, a); (?) \) must be unique, we must have that \( F_P(id_i)(u) = id_{F_P(i)(u)} \). □

Note, that in case of a simple fibration we will only have a natural equivalence between \( F_P(\sigma); F_P(\tau) \) and \( F_P(\sigma \tau) \).

**Example 3.15** (Indexed Categories Spec and Sub)

The above fact says that the assignments \( \phi: \Sigma_1 \rightarrow \Sigma_2 \mapsto Spec_{Pr}(\phi): Spec_{Pr}(\Sigma_1) \rightarrow Spec_{Pr}(\Sigma_2) \) and \( \phi: \Sigma_1 \rightarrow \Sigma_2 \mapsto Mod^{-1}(\phi): Sub_{Pr}(\Sigma_1) \rightarrow Sub_{Pr}(\Sigma_2) \) extends to indexed categories \( Spec_{Pr}: \text{Sign} \rightarrow \text{Cat} \) and \( Sub_{Pr}: \text{Sign} \rightarrow \text{Cat} \), respectively.

It remains to investigate if the equivalence between “indexed categories” and “split opfibrations” extends also to morphisms. Firstly, we have to define

**Definition 3.16** (Maps of Split Opfibrations)

Let \( P: C \rightarrow \text{Ind} \) and \( P': C' \rightarrow \text{Ind} \) be two split opfibrations with splittings \( \kappa \) and \( \kappa' \), respectively. A MAP OF SPLIT OFPIBRATIONS \( \zeta: P \rightarrow P' \) is a functor \( \zeta: C \rightarrow C' \) for which the following conditions are satisfied:

**MSO-1 Compatibility:** The diagram below commutes:

\[
\begin{array}{ccc}
C & \xrightarrow{\zeta} & C' \\
\downarrow P & & \downarrow P' \\
\text{Ind} & & \text{Ind}
\end{array}
\]

**MSO-2 Cleavage preservation:** For any arrow \( \sigma: i \rightarrow j \) in \( \text{Ind} \) and object \( a \) of \( C \) such that \( P(a) = i \), we must have

\[
\zeta(\kappa(\sigma, a)) = \kappa'(\sigma, \zeta(a))
\]

The category of \( \text{Ind} \)-split opfibrations and \( \text{Ind} \)-maps of split opfibrations will be denoted \( \text{SO}(\text{Ind}) \).

Secondly, we can extend the Grothendieck construction to morphisms.
PROPOSITION 3.17
Let \(\text{SO}(\text{Ind})\) and \(\text{IndFun}(\text{Ind}, \text{Cat})\) be respectively, the categories of \(\text{Ind}\)-split opfibrations and \(\text{Ind}\)-indexed functors. Then we can define a functor \(J: \text{IndFun}(\text{Ind}, \text{Cat}) \rightarrow \text{SO}(\text{Ind})\) in the following way:

1. If \(F: \text{Ind} \rightarrow \text{Cat}\) is an indexed category, then \(Pr_F: \text{Flat}(F) \rightarrow \text{Ind}\) is the projection functor of definition 3.1.
2. If \(\alpha: F \Rightarrow G: \text{Ind} \rightarrow \text{Cat}\) is an indexed functor (an arrow in \(\text{IndFun}(\text{Ind}, \text{Cat})\)) then we can define a functor \(\zeta_\alpha: Pr_F \rightarrow Pr_G\) through the following assignments:

   **objects** If \((i, a)\) is an object of \(\text{Flat}(F)\) then \(\zeta_\alpha((i, a)) \overset{\text{def}}{=} (i, \alpha(i)(a))\) is an object of \(\text{Flat}(G)\).

   **arrows** If \((\sigma, f) \rightarrow (j, b)\) (i.e. \(\sigma: i \rightarrow j\) an arrow in \(\text{Ind}\) and \(f: b \rightarrow F(\sigma)(a)\) an arrow in \(\text{Flat}(j)\)), then \(\zeta_\alpha((\sigma, f)) = (\sigma, \alpha(j)(f))\) which is an arrow from \((i, \alpha(i)(a))\) to \((j, \alpha(j)(b))\) (i.e., \(\sigma: i \rightarrow j\) in \(\text{Ind}\) and \(\alpha(j)(f): \alpha(j)(b) \rightarrow \alpha(j)(F(\sigma)(a))\)) in \(G(j)\) and thus an arrow in \(\text{Flat}(G)\).

**Proof.** To show that \(\zeta_\alpha\) is a map of split opfibrations we have to show that it preserves fibers and opcartesian arrows. The first follow directly from definition above. Now let \(\sigma: i \rightarrow j\) in \(\text{Ind}\) and \((i, a)\) an object of \(\text{Flat}(F)\). Now note that \(\zeta_\alpha(\kappa(\sigma, (i, a))) = \zeta_\alpha((\sigma, id_F(\sigma)(a))) = (\sigma, \alpha(j)(id_F(\sigma)(a))) = (\sigma, id_G(\sigma)(\alpha(i)(a))) = \kappa'(\sigma, (i, \alpha(i)(a))).\) Preservation of composition and identities follow from the definition of \(\zeta_\alpha\) and the fact that \(\alpha\) is an indexed functor.

**Example 3.18**
Considering the fact above and given the indexed categories \(\text{Spec}: \text{Ind} \rightarrow \text{Cat}\) and \(\text{Sub}: \text{Ind} \rightarrow \text{Cat}\), together with the natural transformation \(\text{mod}: \text{Spec} \Rightarrow \text{Sub}: \text{Ind} \rightarrow \text{Cat}\), we can define the map of split opfibrations \(\zeta_{\text{mod}}: Pr_{\text{Spec}} \rightarrow Pr_{\text{Sub}}\) through the assignments:

- for \(\langle \Sigma, \Gamma \rangle\) an object of \(\text{Flat}(\text{Spec})\), \(\zeta_{\text{mod}}(\langle \Sigma, \Gamma \rangle) \overset{\text{def}}{=} \langle \Sigma, \text{mod}(\Sigma)(\Gamma) \rangle;\)
- for \(\langle \varphi, \subseteq \rangle: \langle \Sigma_1, \Gamma_1 \rangle \rightarrow \langle \Sigma_2, \Gamma_2 \rangle\) (i.e., \(\sigma: \Sigma_1 \rightarrow \Sigma_2\) and \(\Gamma_2 \supseteq \text{Sen}(\varphi)(\Gamma_1)\)), \(\zeta_{\text{mod}}(\varphi, \subseteq) = \langle \varphi, \subseteq \rangle: \langle \Sigma_1, \text{mod}(\Sigma_1)(\Gamma_1) \rangle \rightarrow \langle \Sigma_2, \text{mod}(\Sigma_2)(\Gamma_2) \rangle\) (that is to say, \(\varphi: \Sigma_1 \rightarrow \Sigma_2\) and \(\text{mod}(\Sigma_2)(\Gamma_2) \subseteq \text{Mod}(\varphi)^{-1}(\text{mod}(\Sigma_1)(\Gamma_1))\)), an arrow in \(\text{Flat}(\text{Sub})\). Note also that \(\zeta_{\alpha}\) preserves the splitting, since given \(\phi: \Sigma_1 \rightarrow \Sigma_2\) in \(\text{Sign}\) and \(\langle \Sigma_1, \Gamma_1 \rangle\) an object of \(\text{Flat}(\text{Spec})\), the opcartesian arrow \(\langle \phi, id_{\Sigma_2}(\Sigma_2)(\Gamma_1) \rangle: \langle \Sigma_1, \Gamma_1 \rangle \rightarrow \langle \Sigma_2, \text{mod}(\Sigma_2)(\Sigma_1)(\Gamma_1) \rangle\) is mapped to \(\langle \phi, \subseteq \rangle: \langle \Sigma_1, \text{mod}(\Sigma_1)(\Gamma_1) \rangle \rightarrow \langle \Sigma_2, \text{mod}(\Sigma_2)(\Sigma_2)(\Gamma_1) \rangle\).

But since \(\text{mod}\) is a natural transformation, we have that \(\text{mod}(\Sigma_2)(\text{Sen}(\varphi)(\Gamma_1)) = \text{Mod}(\varphi)^{-1}(\text{mod}(\Sigma_1)(\Gamma_1))\). Thus \(\zeta_{\text{mod}}(\phi, id_{\text{Sen}(\varphi)(\Gamma_1) \rightarrow \text{mod}(\Sigma_2)(\Sigma_1)(\Gamma_1)}) = \zeta_{\text{mod}}(\phi, \varphi)^{-1}(\text{mod}(\Sigma_1)(\Gamma_1))\), and hence that \(\zeta_{\text{mod}}(\kappa_{\text{Spec}}(\phi, \Gamma_1)) = \zeta_{\text{mod}}(\kappa_{\text{Sub}}(\phi, \Gamma_1))\).

Note that we also have a functor \(\zeta_{\text{th}}: \text{Flat}(\text{Sub}) \rightarrow \text{Flat}(\text{Spec})\), such that for each object \(\langle \Sigma, M \rangle\) of \(\text{Flat}(\text{Sub})\), \(\zeta_{\text{th}}(\langle \Sigma, M \rangle) \overset{\text{def}}{=} \langle \Sigma, \text{th}(M) \rangle\). Moreover, that this functor preserves fibers, in the sense that \(\zeta_{\text{th}}: \text{Pr}_{\text{Spec}} \rightarrow \text{Pr}_{\text{Sub}}\).

Thirdly, the inversion of the Grothendieck construction extends also to morphisms.
Proposition 3.19

Let $SO(\text{Ind})$ and $\text{IndFun}(\text{Ind}, \text{Cat})$ be respectively, the categories of Ind-split opfibrations and Ind-indexed functors. Then we can define a functor $K : SO(\text{Ind}) \to \text{IndFun}(\text{Ind}, \text{Cat})$ in the following way:

1. If $P : C \to \text{Ind}$ is a split opfibration in $SO(\text{Ind})$, then $F_P : \text{Ind} \to \text{Cat}$ is the functor of Fact 3.14.

2. If $\zeta : P \to P'$ is a map of split opfibrations, where $P : C \to \text{Ind}$ and $P' : C' \to \text{Ind}$, then we can define an indexed functor $\alpha_\zeta : F_P \Rightarrow F_{P'} : \text{Ind} \to \text{Cat}$ by restricting $\zeta$ to the fibers of $F_P$, i.e., we assigning to each object $i$ of Ind the restriction $\alpha_\zeta(i) \coloneqq \zeta_{F_P(i)} : F_P(i) \to F_{P'}(i)$.

**Proof.** The commutativity $\zeta_{P'} = P$ ensures $\zeta(F_P(i)) \subseteq F_{P'}(i)$ thus $\alpha_\zeta(i)$ is indeed defined and, as a restriction of the functor $\zeta$, it is obviously a functor too. It remains to show that the functors $\alpha_\zeta(i) = \zeta_{F_P(i)}$ constitute also a natural transformation. We have to show that for each $\sigma : i \to j$ in Ind, the following diagram commutes:

Now let $u : a \to a'$ in $F_P(i)$ and consider the following diagram:

Applying the functor $\zeta$ to the top diagram in the above cube and using the assumption that $\zeta$ is cleavage preserving, we get the following commutative square:

Now we consider for the arrow $\zeta_{F_P(i)}(u)$ in $F_P'(i)$ the following diagram:
object \( \Sigma \) of the corresponding natural transformation be called Frame according to the constructions in subsection 3.1 we obtain structures that will Frame in terms of “fibred categories”. By flattening the components of an Indexed K

The functor Fact 3.21 indeed fully equivalent.

Consider the examples 3.13 and 3.15. Given a map

Example 3.20 Now both \( \hat{u} \) and \( \tilde{u} \) are solutions to the top diagram in the above cube. Thus, by the uniqueness requirement, they are equal and (1) commutes.

Note, that \( F_P(\sigma)(\zeta_{F_P(i)}(a)) = \zeta_{F_P(j)}(F_P(\sigma)(a)) \) because \( \zeta \) is cleavage preserving. Since \( \kappa'(\sigma, \zeta_{F_P(i)}(a)) \) is op-cartesian for \( \sigma \) and \( \zeta_{F_P(i)}(a) \), \( F_P(\sigma)(\zeta_{F_P(i)}(u)) \) is the unique arrow that makes the top square in the above diagram commute. Now let \( \hat{u} \overset{\text{def}}{=} F_P(\sigma)(\zeta_{F_P(i)}(u)) \) and \( \tilde{u} \overset{\text{def}}{=} \zeta_{F_P(j)}(F_P(\sigma)(u)) \) and consider the following calculation:

\[
\begin{align*}
\kappa'(\sigma, \zeta_{F_P(i)}(a')); \hat{u} &= \zeta_{F_P(i)}(u); \kappa'(\sigma, \zeta_{F_P(i)}(a')) \\
&= \zeta_{F_P(i)}(u); \zeta_{F_P(i)}(\kappa(\sigma, a')) \\
&= \zeta(F_P(\sigma); \kappa(\sigma, a')) (\zeta \text{ is a functor}) \\
&= \zeta_{F_P(i)}(\kappa(\sigma, a)); \zeta_{F_P(j)}(F_P(\sigma)(u)) (\zeta \text{ is a functor}) \\
&= \kappa'(\sigma, \zeta_{F_P(i)}(a)); \tilde{u} (\zeta \text{ is a split map})
\end{align*}
\]

Now both \( \hat{u} \) and \( \tilde{u} \) are solutions to the top diagram in the above cube. Thus, by the uniqueness requirement, they are equal and (1) commutes.

**Example 3.20** Consider the examples 3.13 and 3.15. Given a map \( \alpha: \text{Spec}_{F_P} \to \text{Sub}_{F_P} \), we define the corresponding natural transformation \( \zeta: \text{Spec}_{F_P} \Rightarrow \text{Sub}_{F_P} : \text{Ind} \to \text{Cat} \) for each object \( \Sigma \) of \( \text{Sign} \) as \( \alpha_{\zeta}(\Sigma) \overset{\text{def}}{=} \zeta_{\text{Spec}_{F_P}(\Sigma)} \).

Finally, we have that the concepts “indexed category” and “split opfibration” are indeed fully equivalent.

**Fact 3.21** The functor \( K: \text{SO(C)} \to \text{IndFun(Ind, Cat)} \) defined in proposition 3.19 is an equivalence of categories with pseudo-inverse \( J: \text{IndFun(Ind, Cat)} \to \text{SO(Ind)} \) defined in proposition 3.17.

### 3.2 Split Fibred Frames

This subsection is devoted to give an equivalent formulation of the concept of Indexed Frame in terms of “fibred categories”. By flattening the components of an Indexed Frame according to the constructions in subsection 3.1 we obtain structures that will be called
**Definition 3.22 (Split Fibred Frame)**

A **Split Fibred Frame**

\[ \mathcal{SFF} = (\text{Ind}, \text{Syn}, \text{Den}, \text{Fml}, \text{Sem}, \text{P}_{\text{Syn}}, \text{P}_{\text{Den}}) \]

is comprised of the following data and operations:

- A category \( \text{Ind} \).
- A category \( \text{Syn} \) of (abstract) specifications and specification translations.
- A category \( \text{Den} \) of (abstract) model structures and model translations.
- Two split opfibrations \( \text{P}_{\text{Syn}} : \text{Syn} \to \text{Ind} \) with a cleavage \( \kappa_{\text{Syn}} \) and \( \text{P}_{\text{Den}} : \text{Den} \to \text{Ind} \) with a cleavage \( \kappa_{\text{Den}} \).
- A map of split opfibrations \( \text{Sem} : \text{P}_{\text{Syn}} \to \text{P}_{\text{Den}} \), i.e., \( \text{P}_{\text{Syn}} = \text{Sem} ; \text{P}_{\text{Den}} \) such that \( \text{Sem}(\kappa_{\text{Syn}}(\sigma, S)) = \kappa_{\text{Den}}(\sigma, \text{Sem}(S)) \), for any arrow \( \sigma : i \to j \) in \( \text{Ind} \) and object \( S \) of \( \text{Syn} \) such that \( \text{P}_{\text{Syn}}(S) = i \).
- A functor \( \text{Fml} : \text{P}_{\text{Den}} \to \text{P}_{\text{Syn}} \).

Moreover, these operations should satisfy the following properties:

1. The functor \( \text{Fml} \) should be left-adjoint to \( \text{Sem} \), \( \text{Fml} \dashv \text{Sem} \).
2. \( \text{P}_{\text{Den}} = \text{Fml} ; \text{P}_{\text{Syn}} \).

The equivalence between “indexed categories” and “split opfibrations” provides immediately

**Theorem 3.23**

A 5-tuple \( \mathcal{I} = (\text{Ind}, \text{Syn}, \text{Den}, \text{sem}, \text{fml}) \) is an indexed frame if the 7-tuple \( \mathcal{SF}(\mathcal{I}) = (\text{Ind}, \text{Flat(Syn)}, \text{Flat(Den)}, \zeta_{\text{sem}}, \zeta_{\text{fml}}, \text{Pr}_{\text{Syn}}, \text{Pr}_{\text{Den}}) \) is a split fibred frame, called the **Split Fibred Frame** for \( \mathcal{I} \).

And due to Theorem 2.18 we get another equivalent formulation of the concept of Institution

**Corollary 3.24**

A 4-tuple \( \mathcal{I} = (\text{Sign}, \text{Sen}, \text{Mod}, \models) \) is an Institution if the 7-tuple \( \mathcal{SF}(\mathcal{P}(\mathcal{I})) = (\text{Sign}, \text{Flat(Spec)}, \text{Flat(Sub)}, \zeta_{\text{mod}}, \zeta_{\text{th}}, \text{Pr}_{\text{Spec}}, \text{Pr}_{\text{Sub}}) \) is a split fibred frame, called the **Split Fibred Institution** for \( \mathcal{I} \).

Now, we went to analyse the flattening of Maps of Indexed Frames. A crucial technical observation is that the flattening of the indexed functors involved in the definition of Maps of Indexed Frames provides in addition to the Map of Split Opfibrations according to Proposition 3.17 also a pullback diagram describing the “re-typing” induced by a transformation of indices.
Proposition 3.25
Let $\Psi : \text{Ind} \to \text{Ind}'$, $F : \text{Ind} \to \text{Cat}$, and $G : \text{Ind}' \to \text{Cat}$ be functors. For any indexed functor $\alpha : F \Rightarrow \Psi; G : \text{Ind} \to \text{Cat}$ the assignments

- $\Psi G(i, a) = (\Psi(i), a)$ for any $(i, a)$ in $\text{Flat}(\Psi; G)$
- $\Psi G(\sigma, f) = (\Psi(\sigma), f)$ for any $(\sigma, f)$ in $\text{Flat}(\Psi; G)$

define a functor $\Psi G : \text{Flat}(\Psi; G) \to \text{Flat}(G)$ such that the following diagram is a pullback diagram in $\text{Cat}$.

\[
\begin{array}{ccc}
\text{Ind} & \xrightarrow{\Psi} & \text{Ind}' \\
\downarrow & & \downarrow \Pr_{\Psi,G} \\
\text{Flat}(\Psi; G) & \xrightarrow{\Psi G} & \text{Flat}(G)
\end{array}
\]

In view of the Propositions 3.17, 3.25 and the intended equivalence we have to define a kind of complicated and technical concept of

Definition 3.26 (Split Maps of Split Fibred Frames)
Let

$\mathcal{SSF} = (\text{Ind}, \text{Syn}, \text{Den}, \text{Fml}, \text{Sem}, P_{\text{Syn}}, P_{\text{Den}})$

and

$\mathcal{SSF}' = (\text{Ind}', \text{Syn}', \text{Den}', \text{Fml}', \text{Sem}', P_{\text{Syn}'}, P_{\text{Den}'})$

be two split fibred frames. A split map of split fibred frames $\pi : \mathcal{SSF} \to \mathcal{SSF}'$ is a triple $\pi = (\Phi, \Psi_{\text{Syn}}, \Psi_{\text{Den}})$ with

- a functor $\Phi : \text{Ind} \to \text{Ind}'$;
- a functor $\Psi_{\text{Syn}} : \text{Syn} \to \text{Syn}'$;
- a functor $\Psi_{\text{Den}} : \text{Den} \to \text{Den}'$;

such that the following properties are satisfied:

1. $\text{Sem}; \Psi_{\text{Den}} = \Psi_{\text{Syn}}; \text{Sem}'$;
2. $\Psi_{\text{Syn}}(\kappa_{\text{Syn}}(\sigma, S)) = \kappa_{\text{Syn}'}(\Phi(\sigma), \Psi_{\text{Syn}}(S))$ for every $\sigma : i \to j$ in $\text{Ind}$ and every object $S$ of $\text{Syn}$ such that $P_{\text{Syn}}(S) = i$;
3. $\Psi_{\text{Den}}(\kappa_{\text{Den}}(\sigma, D)) = \kappa_{\text{Den}'}(\Phi(\sigma), \Psi_{\text{Den}}(D))$ for every $\sigma : i \to j$ in $\text{Ind}$ and every object $D$ of $\text{Den}$ such that $P_{\text{Den}}(D) = i$;
4. For $\Psi_{\text{Syn}} : \text{Syn} \to \text{Syn}'$ there exists a split obfibration $P_{\text{Syn}'}[\Phi] : \text{Syn}'[\Phi] \to \text{Ind}$ and functors $\zeta_{\text{Syn}} : \text{Syn} \to \text{Syn}'[\Phi]$, $\Phi_{\text{Syn}} : \text{Syn}'[\Phi] \to \text{Syn}'$ such that $\Psi_{\text{Syn}} = \zeta_{\text{Syn}}; \Phi_{\text{Syn}}$ and $\zeta_{\text{Syn}} : P_{\text{Syn}'}[\Phi] \to P_{\text{Syn}'}[\Phi]$ is a map of split obfibrations, and the following square is a pullback diagram, i.e., we have also $\Psi_{\text{Syn}}; P_{\text{Syn}'} = P_{\text{Syn}}; \Phi$.

\[
\begin{array}{cccc}
\text{Ind} & \xrightarrow{\Phi} & \text{Ind}' \\
\downarrow & & \downarrow \Pr_{\Phi} \\
\text{Syn} & \xrightarrow{\zeta_{\text{Syn}}} & \text{Syn}'[\Phi] & \xrightarrow{\Phi_{\text{Syn}}} \text{Syn}'
\end{array}
\]
For \( \Psi_{Den} : Den \to Den' \) there exists a split fibration \( P_{Den}[\Phi] : Den'[\Phi] \to \text{Ind} \) and functors \( \zeta_{Den} : Den \to Den'[\Phi], \Phi_{Den} : Den'[\Phi] \to Den' \) such that \( \Psi_{Den} = \zeta_{Den}; \Phi_{Den}, \zeta_{Den} : P_{Den'} \to P_{Den}[\Phi] \) is a map of split fibrations, and the following square is a pullback diagram i.e., we have also \( \Psi_{Den}; P_{Den'} = P_{Den'}; \Phi, \) \[ \begin{array}{c}
\text{Ind} \\
\downarrow P_{Den}[\Phi] \\
\Phi \\
\Phi_{Den} \\
\downarrow P_{Den'} \\
\text{Ind}' \\
\end{array} \]

The usual component wise definition of composition provides the

**Definition 3.27 (Category of Split Fibred Frames)**

Given split maps of split fibred frames \( \pi : (\Phi, \Psi_{Syn}, \Psi_{Den}) : \text{SFF} \to \text{SFF}' \) and \( \omega : (\Phi', \Psi_{Syn}', \Psi_{Den}') : \text{SFF}' \to \text{SFF}'' \) we define their composition as

\[ \pi; \omega : (\Phi; \Phi', \Psi_{Syn}; \Psi_{Syn}', \Psi_{Den}; \Psi_{Den}') : \text{SFF} \to \text{SFF}'. \]

The category where the objects are split fibred frames and the arrows are maps of split fibred frames is called \text{SplitFram}.

Immediately by Proposition 3.17 and 3.25 we obtain

**Theorem 3.28**

A triple \( \xi = (\Upsilon, \gamma, \delta) : \text{IF} \to \text{IF}' \) is a map of Indexed Frames if and only if the triple \( SF(\xi) = (\Phi, \zeta_{\gamma}; \Upsilon_{Den}', \zeta_{\delta}; \Upsilon_{Syn}') : SF(\text{IF}) \to SF(\text{IF}') \) is a split map of split fibred frames.

Based on the Theorems 3.23 3.28 we obtain

**Proposition 3.29**

Let \text{IndFra} and \text{SplitFra} be, respectively, the categories of Indexed Frames and Split Fibred Frames. Then, there is a functor \( SF : \text{IndFra} \to \text{SplitFra} \) which is defined in the following way:

1. Given an indexed frame \( \text{IF} = (\text{Ind}, \text{Syn}, \text{Den}, \text{sem}, \text{fml}) \) the corresponding split fibred frame \( SF(\text{IF}) \) is defined by

\[ SF(\text{IF}) \overset{def}{=} (\text{Ind}, \text{Flat(Syn)}, \text{Flat(Den)}, \zeta_{\text{sem}}, \zeta_{\text{fml}}, \text{Pr}_{\text{Syn}}, \text{Pr}_{\text{Den}}) \]

according to Theorem 3.23.

2. Given a map of indexed frames \( \xi = (\Upsilon, \gamma, \delta) : \text{IF} \to \text{IF}' \) the corresponding split map of split fibred frames is defined by

\[ SF(\xi) \overset{def}{=} (\Phi, \zeta_{\gamma}; \Upsilon_{\text{Den}'}, \zeta_{\delta}; \Upsilon_{\text{Syn}'} : SF(\text{IF}) \to SF(\text{IF}') \]

according to Theorem 3.28.
Now, we are ready to define the complete translation of Split Fibred Frames into Indexed Frames.

**Proposition 3.30**

Let IndFra and SplitFra be, respectively, the categories of Indexed Frames and Split Fibred Frames. Then, there is a functor \( M : \text{SplitFra} \to \text{IndFra} \) which is defined in the following way:

1. Given a split fibred frame \( SFF = (\text{Ind}, \text{Syn}, \text{Den}, F_{\text{Fml}}, \text{Sem}, P_{\text{Syn}}, P_{\text{Den}}) \), we define the corresponding indexed frame \( M(SFF) = (\text{Ind}, F_{P_{\text{Syn}}}, F_{P_{\text{Den}}}, \alpha_{\text{Sem}}, \alpha_{F\text{ml}}) \), where
   - \( F_{P_{\text{Syn}}} : \text{Ind} \to \text{Cat} \) and \( F_{P_{\text{Den}}} : \text{Ind} \to \text{Cat} \) are indexed categories according to Fact 3.14;
   - \( \alpha_{\text{Sem}} : F_{P_{\text{Syn}}} \Rightarrow F_{P_{\text{Den}}} : \text{Ind} \to \text{Cat} \) is an indexed functor according to Proposition 3.19, i.e., for each \( i \in \text{Ind} \) the functor \( \alpha_{\text{Sem}}(i) : F_{P_{\text{Syn}}}(i) \to F_{P_{\text{Den}}}(i) \) is the restriction of \( \text{Sem} : \text{Syn} \to \text{Den} \) to \( P_{\text{Syn}}^{-1}(\text{id}_i) \); and
   - \( \alpha_{F\text{ml}} : F_{P_{\text{Den}}}(i) \Rightarrow F_{P_{\text{Syn}}}(i) \) the restriction of \( F_{\text{Fml}} : \text{Den} \to \text{Syn} \) to \( F_{P_{\text{Den}}}(i) = P_{\text{Den}}^{-1}(\text{id}_i) \) for each \( i \in \text{Ind} \).
2. Given a map of split fibred frames \( \pi = (\Phi, \Psi_{\text{Syn}}, \Psi_{\text{Den}}) : SFF \to SFF' \), we define the corresponding map of indexed frames \( M(\pi) : M(SFF) \to M(SFF') \) by \( M(\pi) \overset{\text{def}}{=} (\Phi, \alpha_{\text{Den}}, \alpha_{\text{Syn}}) \) according to Proposition 3.19.

Finally, we obtain the intended complete axiomatization of Indexed Frames in terms of “fibred categories”.

**Theorem 3.31**

The functor \( SF : \text{IndFra} \to \text{SplitFra} \) is an equivalence of categories with pseudo-inverse \( M : \text{SplitFra} \to \text{IndFra} \) defined in proposition 3.30.

### 4 Conclusion and Future Work

In this paper we have addressed the question “What essential mathematical structures underly logical systems with indexed syntax and semantics on the level of abstraction given by specifications and model classes?”. We provided an indexed general definition of Logics that takes indices as the guideline for building every linguistically relevant concept in the language. The proposed concept of **Indexed Frame** appears as an abstraction of the concept of **Institution** and provides, especially, an elegant and natural account of the “institution condition”. The relevance of the chosen level of abstraction is also validated by the fact that most applications of Institutions within the theory of Formal Specifications focus on specifications and model classes.

In a forthcoming paper [14] we show that other well-known definitions of Abstract Logical Frameworks, as \( \pi \)-institutions, Entailment Systems, and thus General Logics can be also reflected by Indexed Frames.

It is worth to mention that our approach, being based on adjunctions, is somehow related to Hyperdoctrines, the structure that captures predicates and all logical operations as adjunctions [16]. The approach presented here follows Lawvere’s idea of regarding adjunctions as one of the most fundamental concepts in logic. The main difference between both approaches is that while Hyperdoctrines work out the adjunc-
In the second part of the paper we have investigated how the concept of Indexed Frame can be equivalently expressed in terms of “fibred categories”. In contrast to the clean and simple concept of Indexed Frame the equivalent concept of Split Fibred Frame turned out to be very technical. This fact sheds some light on the insight that the presentation of Logics by means of Institutions or Indexed Frames, respectively, is only possible if a lot of compatibility requirements are satisfied which are often hard to meet by our concrete syntactical constructions of Logics. Therefore an indexed presentation of a Logic has to be based on a heavy use of the “axiom of choice” or on a 2-categorical framework extending the framework presented in this paper (compare [8, 9, 22]).

Instead of using a complicated and more technical 2-categorical framework it seems to be more natural, in accordance with Bénabou’s foundational article [4] and with [15], to work with an equivalent clean and simple concept of “fibred frames” obtained by dropping the different “split requirements” in the definition of Split Fibred Frames. The exploration of these possibilities will be one topic of future research.

Closure Operators and Closure Systems, respectively, are well-known and well-accepted concepts for describing and investigating Logics. Closure Operators are implicitly present in our framework as the (co)monads of the adjunctions $\text{fml}((i) \dashv \text{sem}((i))$. The development of an appropriate theory of “indexed and fibred closure operators” will be another topic of future research.

Finally, it seems to be worth to investigate, in detail, the relation of our framework to the framework of Grothendieck Institution [8, 9, 22] and of Fibred Institution [9].

References

Fibred and Indexed Categories for Abstract Model Theory


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